Non-uniqueness of generalized Navier-Stokes equations in subcritical spaces

Zipeng Chen[1] *, Song Liu[1] † and Zhaoyang Yin[1‡]
[1]School of Science,
Shenzhen Campus of Sun Yat-sen University, Shenzhen 518107, China

Abstract

In this paper, we consider the generalized Navier-Stokes equations with fritional dissipation $(-\Delta)^{\beta}$ with $\beta > \frac{1}{2}$. When $\beta \in (1,2)$, We prove that smooth solutions of the generalized Navier-Stokes equations are non-unique with arbitrarily small initial data in $\dot{B}^{-\beta-\alpha}_{\infty,1}(\mathbb{T}^d)$ for any $\alpha > 0$. It is worth pointing out that the space $\dot{B}^{-\beta-\alpha}_{\infty,1}(\mathbb{T}^d)$ is subcritical for $0 < \alpha < \beta - 1$. To the best of our knowledge, this is the first non-uniqueness result of Navier-Stokes equations with initial data at the critical regularity. To show the sharpness of the above results, for $\beta > \frac{1}{2}$, we establish the local well-poseness of the generalized Navier-Stokes equations with small initial data in $\dot{B}^{-\beta-\alpha}_{\infty,\infty}(\mathbb{T}^d)$ with $\alpha < 0$ and $\alpha \leq \beta - 1$.

Keywords: , Non-uniqueness, Navier-Stokes equations,

AMS Subject Classification: 35Q30, 76D03

Contents

1 Introduction **2**
- 1.1 Main result . . . 3
- 1.2 Main idea of the construction . . . 4
- 1.3 Notation and parameters . . . 5
- 1.4 Organization of the paper . . . 7

2 The leading part of the solutions **7**
- 2.1 Construction of the initial data . . . 7
- 2.2 Construction of the principal parts . . . 11
- 2.3 Estimates on the residual . . . 13

*Email: chenzp26@mail2.sysu.edu.cn
†Email: lius37@mail2.sysu.edu.cn
‡E-mail: mcsyzy@mail.sysu.edu.cn

3 Construction of the perturbation **18**
3.1 Semigroup of the linearization around the principal part 18
3.2 Fixed point argument 20

4 Proof of the main results **21**

A Tools from convex integration **24**

B Fractional Leibniz estimate **24**

C Grönwall inequality and heat esitmate **26**

References **27**

1 Introduction

In this paper, we consider the following d-dimensional generalized Navier-Stokes equations on $\mathbb{T}^d$:

$$\begin{cases} \partial_t v + \operatorname{div}(v \otimes v) + (-\Delta)^\beta v + \nabla p = 0, \\ \operatorname{div}\ v = 0, \qquad \forall (t,x) \in [0,1] \times \mathbb{T}^d \end{cases} \tag{1.1}$$

where $d \geq 2$, $\beta > \frac{1}{2}$ and $\mathbb{T}^d$ is the d-dimensional toru. Here, v and p represent the flow velocity and the scalar pressure, respectively.

When $\beta = 1$, system (1.1) takes the form of the famous incompressible Navier–Stokes equations:

$$\begin{cases} \partial_t v + \operatorname{div}(v \otimes v) - \Delta v + \nabla p = 0, \\ \operatorname{div} v = 0, \end{cases} \tag{1.2}$$

In the remarkable paper [25], for any dimension $d \geq 2$, Leray first showed that there exists a weak solution in $L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^d)) \cap L^2(\mathbb{R}^+; \dot{H}^1(\mathbb{R}^d))$ for any L^2 solenoidal initial data, which satisfies the energy inequality

$$\|v(t)\|_{L^2}^2 + 2\int_0^t \|\nabla v(s)\|_{L^2}^2 ds \leq \|v(0)\|_{L^2}^2, \quad \forall t \geq 0.$$

For smooth bounded domains with Dirichlet boundary conditions, Hopf [17] derived an analogous result. This class of weak solutions is now known as Leray–Hopf weak solutions. To this day, the uniqueness problem for Leray–Hopf weak solutions of the Navier–Stokes equations remains an open question in dimensions greater than two.

By the mild formulation of equations, the global well-posedness for small data and local well-posedness for large data are derived in some critical space

$$H^{\frac{d}{2}-1} \subset L^d \subset \dot{B}^{-1+\frac{d}{p}}_{p,\infty} \subset BMO^{-1} \subset \dot{B}^{-1}_{\infty,\infty}$$

where $d < p < \infty$. These norms are all invariant under the following scaling transformations:

$$v(t,x) \mapsto \lambda v(\lambda^2 t, \lambda x), \quad p(t,x) \mapsto \lambda^2 p(\lambda^2 t, \lambda x).$$

Fujita-Kato[14] and Kato[21] proved that this holds for initial data in the first two spaces, respectively. A corresponding result in Besov spaces $B^{-1+\frac{d}{p}}_{p,\infty}(\mathbb{R}^d)(1 \le q < \infty)$ was obtained by Cannone [7] and Planchon [30]. Koch-Tataru[22] established that global well-posedness holds for small data in BMO^{-1}. The weak-strong uniqueness property for (1.2) in critical spaces has also attracted considerable attention in the literature. We refer the reader to the references [23, 24, 31, 32] on NS equations.

Nevertheless, ill-posedness or non-uniqueness will also arise in the critical class. For the largest critical space $B^{-1}_{\infty,\infty}$, Bourgain-Pavlović [3] showed that norm inflation is possible in 3D Navier-Stokes equations. In other words, the solutions with arbitrarily small data in $B^{-1}_{\infty,\infty}$ can grow arbitrarily large in a short time. On the other hand, Germain [15] proved that the data-to-solution map is not C^2 in the spaces $B^{-1}_{\infty,q}$ for $q > 2$ and Yoneda [36] showed that ill-posedness holds in this class. Later in [35], Wang showed that norm inflation from small data can occur even in the spaces $B^{-1}_{\infty,q}$ for $q \in [1, 2]$, which are continuously embedded in BMO^{-1}. Recently in [10], Coiculescu-Palasek firstly constructed two distinct global solutions of 3D Navier-Stokes equations with identical initial data in space BMO^{-1}. Subsequently, Miao-Nie-Ye [28] obtain the same result for the two-dimensional case.

Driven by the development of convex integration techniques [11, 12, 18, 5, 16], many non-uniqueness results have been obtained in supercritical spaces. The first such result is given by Buckmaster-Vicol [6]. By the convex integration scheme and the L^2_x-based intermittent spatial building block, they showed non-uniqueness of finite energy weak solutions to the 3D Navier-Stokes equations. See the surveys [27, 4] for the fractional dissipation case. Later in [8], Cheskidov-Luo used the temporal intermittency method and proved the non-uniqueness of (1.2) on $L^p_t L^\infty_x$ when $p < 2$. Moreover, Cheskidov-Luo [9] established the non-uniqueness on the class $L^\infty_t L^q_x$ when $q < 2$ for the 2D NS equation. For the 3D hyperdissipative NS equation with hyperviscosity exponent beyond the Lion exponent, Li-Qu-Zeng-Zhang [26] showed non-uniqueness in some supercritical spaces. The above three results are all sharp with respect to the Ladyženskaja-Prodi-Serrin criteria and posed some additional favorable properties outside of singular times with arbitrarily Hausdorff dimension.

We also refer to another programme by Jia and Šverák [19, 20]. They demonstrated that non-uniqueness of Leray–Hopf weak solutions is valid, provided that a suitable spectral condition holds for the linearized Navier–Stokes operator. Inspired by [33, 34], Albritton-Brué-Colombo [1] proved non-uniqueness in the Leray-Hopf weak solutions for the forced 3D Navier-Stokes equations. See also the related work [2] for the hypodissipative Navier-Stokes equations in two dimensions.

1.1. Main result

In [10], the authors demonstrated that uniqueness of the 3D Navier-Stokes equation may break down if the initial data is large in BMO^{-1}. Inspired by their work, we establish non-uniqueness for the generalized Navier-Stokes equation with arbitrarily small initial data. More precisely, we have

Theorem 1.1. *Let $d \ge 2$, $1 < \beta < 2$ and $\alpha > 0$. For any $\varepsilon > 0$ and $0 < p < \frac{2\beta}{\beta+\alpha}$, there exist divergence-free initial data $V^0 \in B^{-\beta-\alpha}_{\infty,1}(\mathbb{T}^d)$ such that $\|V^0\|_{B^{-\beta-\alpha}_{\infty,1}(\mathbb{T}^d)} < \varepsilon$ and the generalized Navier-Stokes equation (1.1) admits two distinct solutions*

$$\begin{aligned} v^{(1)}, v^{(2)} \in & C^\infty_{t,x}((0,1] \times \mathbb{T}^d) \cap C^0([0,1]; B^{-\beta-\alpha}_{\infty,1}(\mathbb{T}^d)) \\ & \cap L^p([0,1]; L^\infty(\mathbb{T}^d)). \end{aligned}$$

Theorem 1.2. *Let $\beta > \frac{1}{2}$ and $\alpha \leq \beta - 1$ with $\alpha < 0$, there exists $\varepsilon_\beta > 0$ such that for all $v_0 \in \dot{B}^{-\beta-\alpha}_{\infty,\infty}(\mathbb{T}^d)$ with $\nabla \cdot v_0 = 0$ and*

$$\|v_0\|_{\dot{B}^{-\beta-\alpha}_{\infty,\infty}(\mathbb{T}^d)} \leq \varepsilon_\beta,$$

for (1.1), there exists a solution $v \in L^\infty\big((0,1], \dot{B}^{-\beta-\alpha}_{\infty,\infty}(\mathbb{T}^d)\big)$ such that

$$\sup_{t\in(0,1]} t^{\frac{\beta+\alpha}{2\beta}} \|v(t)\|_{L^\infty(\mathbb{R}^n)} < \infty.$$

Since the solutions in the lower dimensions satisfy the higher-dimensional generalized Navier-Stokes equations when zero components are added to the remaining dimensions. Moreover, the embedding $B^a_{\infty,\infty} \hookrightarrow B^b_{\infty,1}$ holds for all $a > b$. Consequently, proving Theorem 1.1 reduces to establishing the following propositions.

Proposition 1.3. *Let $d = 2$, $1 < \beta < 2$ and $0 < \alpha \leq \beta - 1$ with $\alpha < \frac{2-\beta}{3}$. For any $\varepsilon, \varepsilon' > 0$ and $0 < p < \frac{2\beta}{\beta+\alpha}$, there exist divergence-free initial data $V^0 \in B^{-\beta-\alpha}_{\infty,\infty}(\mathbb{T}^2)$ such that $\|V^0\|_{B^{-\beta-\alpha}_{\infty,\infty}(\mathbb{T}^2)} < \varepsilon$ and the generalized Navier-Stokes equation (1.1) admit two distinct solutions*

$$\begin{aligned} v^{(1)}, v^{(2)} \in & C^\infty_{t,x}((0,1] \times \mathbb{T}^2) \cap C^0([0,1]; B^{-\beta-\alpha-\varepsilon'}_{\infty,\infty}(\mathbb{T}^2)) \\ & \cap L^p([0,1]; L^\infty(\mathbb{T}^2)). \end{aligned}$$

1.2. Main idea of the construction

The non-uniqueness mechanism of the generalized Navier-Stokes equations follows from the approach of [29] and [10]. In this part, we outline the main idea of the proof, omitting certain details for the sake of exposition. In our construction, there exist two types of flows:

• The first is the heat-dominated flow, which is the solution of heat equaiton with fritional dissipation, up to a small error.

$$\partial_t v_k + (-\Delta)^\beta v_k = l.o.t.$$

where $l.o.t$ means some lower order terms. And the heat-dominated flows satisfy the following exponential decay in time:

$$\|v_k\|_{L^\infty(\mathbb{T}^2)} \lesssim N_k^{\beta+\alpha} e^{-N_k^2 t}.$$

• The second is the inverse cascade-dominated flow, whose temporal partial derivative annihilates the nonlinear self-interactions of the heat-dominated flow.

$$\partial_t \bar{v}_k + \mathbb{P}\,\mathrm{div}(v_{k+1} \otimes v_{k+1}) = l.o.t.$$

Compared to the heat-dominated flows, the inverse cascade-dominated flows have slower exponential decay in time:

$$\|\bar{v}_k\|_{L^\infty(\mathbb{T}^2)} \lesssim N_k^{\beta+\alpha} e^{-N_{k+1}^2 t}.$$

The principal part of the solutions to the generalized Navier-Stokes equations is derived and given as follows:

$$v^{(1)} \triangleq \sum_{k\geq 0\, even} v_k + \sum_{k\geq 0\, odd} \bar{v}_k,$$

and

$$v^{(2)} \triangleq \sum_{k\geq 0\, odd} v_k + \sum_{k\geq 0\, even} \bar{v}_k.$$

In the following discussion, i denotes either 1 or 2. As a result of the cancellation between the heat-dominated flows and the inverse cascade-dominated flows, we can deduce remaining terms $F^{(i)}$ which are defined by

$$\partial_t v^{(i)} + (-\Delta)^\beta v^{(i)} + \mathbb{P}\operatorname{div}(v^{(i)} \otimes v^{(i)}) = -\mathbb{P}\operatorname{div} F^{(i)},$$

are arbitrarily small in some subcritial norms. In other words, we conclude that $v^{(1)}$ and $v^{(2)}$ are solutions to the generalized Navier-Stokes equations (1.1) with a small error in divergence form.

In the final step, we add the perturbation $\omega^{(i)}$ to $v^{(i)}$ to correct the associated errors, so that $v^{(i)} + \omega^{(i)}$ is indeed a solution to the generalized Navier-Stokes equations (1.1). Then $\omega^{(i)}$ must satisfy the generalized Navier-Stokes equations

$$\partial_t \omega^{(i)} + (-\Delta)^\beta \omega^{(i)} + \mathbb{P}\operatorname{div}\left(v^{(i)} \otimes \omega^{(i)} + \omega^{(i)} \otimes v^{(i)} + \omega^{(i)} \otimes \omega^{(i)}\right) = \mathbb{P}\operatorname{div} F^{(i)},$$

Given that $F^{(i)}$ is small in certain subcritical spaces, we are able to establish the existence of $\omega^{(i)}$ and confirm its smallness in appropriate subcritical norms by means of a fixed point argument.

Using the fact that $\bar{v}_k(0,\cdot) = v_k(0,\cdot)$, $\bar{b}_k(0,\cdot) = b_k(0,\cdot)$ and $\omega^{(i)}(0,\cdot) = 0$, we know that $v^{(1)} + \omega^{(1)}$ and $v^{(2)} + \omega^{(2)}$ have the same initial data in $B^{-\beta-\alpha}_{\infty,\infty}(\mathbb{T}^2)$. However, owing to the distinct temporal decay behaviors of v_k and $\bar{v}_k$, we can infer that $v^{(1)} + \omega^{(1)}$ and $v^{(2)} + \omega^{(2)}$ are distinct (see details in Section 4).

1.3. Notation and parameters

Suppose $\varphi : \mathbb{R}^d \to [0,1]$ and $\chi : \mathbb{R}^d \to [0,1]$ are two smooth radial functions satisfying

$$\varphi(x) = \begin{cases} 1, \forall |x| \in [\frac{6}{7}, \frac{12}{7}] \\ 0, \forall |x| \in [0,\infty) \setminus [\frac{3}{4}, \frac{8}{3}] \end{cases} \qquad \text{and} \qquad \chi(x) = \begin{cases} 1, \forall |x| \in [0, \frac{3}{4}] \\ 0, \forall |x| \in [0, \frac{4}{3}) \end{cases},$$

and

$$\chi(x) + \sum_{j\in\mathbb{N}} \varphi(2^{-j}x) = 1, \quad \forall x \in \mathbb{R}^d,$$
$$\sum_{j\in\mathbb{Z}} \varphi(2^{-j}x) = 1, \quad \forall x \in \mathbb{R}^d \setminus \{0\}$$

Let $D(\mathbb{T}^d)$ denote the space of infinitely smooth functions on $\mathbb{T}^d$ and $D'(\mathbb{T}^d)$ be defined as its topological dual. Every distribution $f \in D'(\mathbb{T}^d)$ admits the formal Fourier series

$$f = \sum_{l\in\mathbb{Z}^d} \hat{f}(l) e^{2\pi \mathrm{i} l\cdot x},$$

where

$$\hat{f}(l) = \fint_{\mathbb{T}^d} f(x) e^{-2\pi i x\cdot l} dx.$$

The nonhomogeneous dyadic blocks $\{\Delta_j\}_{j\geq -1}$ are defined by

$$\Delta_{-1} f \triangleq \sum_{l\in\mathbb{Z}^d} \chi(l)\widehat{f}(l)e^{2\pi \mathrm{i} l\cdot x}.$$

and for any $j \geq 0$,

$$\Delta_j f \triangleq \sum_{l\in\mathbb{Z}^d} \varphi(2^{-j}l)\widehat{f}(l)e^{2\pi \mathrm{i} l\cdot x}.$$

The homogeneous dyadic blocks $\{\dot{\Delta}_j\}_{j\in\mathbb{Z}}$ are defined for all $j \in \mathbb{Z}$ by

$$\dot{\Delta}_j f \triangleq \sum_{l\in\mathbb{Z}^d} \varphi(2^{-j}l)\widehat{f}(l)e^{2\pi \mathrm{i} l\cdot x}.$$

Definition 1.4. *Let $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$. The nonhomogeneous Besov space $B^s_{p,q}(\mathbb{T}^d)$ consists of all $\mathbb{T}^d$-periodic functions $u \in D'(\mathbb{T}^d)$ such that*

$$\begin{aligned}
\|u\|_{B^s_{p,q}(\mathbb{T}^d)} &\triangleq \left(\sum_{j\geq -1}\left(2^{js}\|\Delta_j u\|_{L^p(\mathbb{T}^d)}\right)^q\right)^{\frac{1}{q}} < \infty, && \text{if} \quad q<\infty,\\
\|u\|_{B^s_{p,\infty}(\mathbb{T}^d)} &\triangleq \sup_{j\geq -1} 2^{js}\|\Delta_j u\|_{L^p(\mathbb{T}^d)} < \infty, && \text{if} \quad q=\infty.
\end{aligned}$$

And the homogeneous Besov space $\dot{B}^s_{p,q}(\mathbb{T}^d)$ consists of all $\mathbb{T}^d$-periodic functions $u \in D'(\mathbb{T}^d)$ such that

$$\begin{aligned}
\|u\|_{\dot{B}^s_{p,q}(\mathbb{T}^d)} &\triangleq \left(\sum_{j\in\mathbb{Z}}\left(2^{js}\|\Delta_j u\|_{L^p(\mathbb{T}^d)}\right)^q\right)^{\frac{1}{q}} < \infty, && \text{if} \quad q<\infty,\\
\|u\|_{\dot{B}^s_{p,\infty}(\mathbb{T}^d)} &\triangleq \sup_{j\in\mathbb{Z}} 2^{js}\|\Delta_j u\|_{L^p(\mathbb{T}^d)} < \infty, && \text{if} \quad q=\infty.
\end{aligned}$$

In this section, we collect the notation used throughout the paper and introduce the parameters appearing in our construction.

We denote $\mathbb{P}$ as the Helmholtz-Weyl projection onto divergence free vector fields

$$\mathbb{P} = \{\mathbb{P}_{j,k}\}_{j,k=1,\dots,d} = \{\delta_{j,k} + R_j R_k\}_{j,k=1,\dots,d}$$

with $\delta_{j,k}$ is the Kronecker symbol and $R_j = \partial_j(-\Delta)^{-\frac{1}{2}}$ are the Riesz transforms.

Let $S^{2\times 2}(\mathbb{R})$ be the space of real symmetric 2×2 matrices. For any real numbers a and b, we write $a \wedge b$ and $a \vee b$ to denote $\min\{a, b\}$ and $\max\{a, b\}$, respectively.

We use the notation $X \lesssim Y$ to mean that $X \leq CY$ with some constants $C > 0$ independent of A, but which may depend on other parameters.

To measure the amplitude and frequency of the building blocks, we introduce the parameters

$$N_k \triangleq A^{b^k} \quad \text{for } k \in \mathbb{N},$$

where A and b belong to $\mathbb{N}$. Finally, $A \gg 1$ depending on other parameters will be chosen sufficiently large in the end.

1.4. Organization of the paper

The organization of the rest of the paper is as follows. In Section 2, we design the initial data and the leading parts of the solutions and verify that the remaining terms are small in some subcritical norms. In Section 3, we estimate the perturbations. In Section 4, we complete the proof of Proposition 1.3 and Theorem 1.2 to show the non-uniqueness and the local well-poseness. Appendix contains some technical tools used in the paper.

2 The leading part of the solutions

Throughout the rest of this paper, we focus on proving Proposition 1.3. Thus, we fix $d = 2$, $1 < \beta < 2$, $\varepsilon > 0$ and $0 < \alpha \leq \beta - 1$ with $\alpha < \frac{2-\beta}{3}$.

2.1. Construction of the initial data

Definition 2.1. *We define*

- *A differential operator S that takes vector fields to the symmetric rank-2 tensor,*

$$S(f) \triangleq \begin{pmatrix} 2\partial_1 f^1 & \partial_1 f^2 + \partial_2 f^1 \\ \partial_1 f^2 + \partial_2 f^1 & 2\partial_2 f^2 \end{pmatrix}, \quad \forall f \in C^\infty(\mathbb{T}^2; \mathbb{R}^2),$$

  Moreover, if f is divergence-free, we have

$$\mathrm{div} S(f) = \Delta f. \tag{2.1}$$

- *A standard mollifier ϕ_k with length scale $N_{k+1}^{-\frac{1}{2}} N_k^{-\frac{1}{2}}$, supported in $B(0,1)$ with $\int_{\mathbb{T}^2} \phi_k = 1$.*

We construct the vector potentials by induction. Recall that the vector set $\{\xi\}_{\xi\in\Lambda}$, the corresponding orthonormal vetors $\{\bar{\xi}\}_{\xi\in\Lambda}$ and the functions $\{a_\xi\}_{\xi\in\Lambda}$ are introduced in Lemma A.1.

Definition 2.2. *Let $\bar{\xi}$ be a vector in Λ. For the zeroth, we define*

$$\Phi_0^0(x) \triangleq N_0^{-2+\beta+\alpha} \varepsilon \sin(2\pi N_0 \xi \cdot x) \bar{\xi}, \tag{2.2}$$

Given a divergence-free Φ_{k-1}^0, we define the vector potential for the k-th case:

$$\Phi_k^0(x) \triangleq N_k^{-3+\beta+\alpha} \varepsilon \sqrt{2} (2\pi)^{\beta-3} i \nabla^\perp \left(\sum_{\xi\in\Lambda} a_{\xi,k} e^{2\pi i N_k \xi \cdot x} \right) * \phi_k(x), \tag{2.3}$$

where

$$a_{\xi,k} \triangleq a_\xi \left(\frac{S(\Phi_{k-1}^0)}{N_k^{2\alpha} \varepsilon^2} + \mathrm{Id} \right). \tag{2.4}$$

To verify that $\{\Phi_k^0\}_{k\in\mathbb{N}}$ are well-defined, it suffices to establish that $\|\frac{S(\Phi_{k-1}^0)}{N_k^{2\alpha}\varepsilon^2}\|_{L^\infty(\mathbb{T}^2)} \leq \frac{1}{7}$ by Lemma A.1.

Lemma 2.3. *There exists a universal constant $C > 0$ such that for all $k \in \mathbb{N}$,*

$$\|\frac{S(\Phi_k^0)}{N_{k+1}^{2\alpha}\varepsilon^2}\|_{L^\infty(\mathbb{T}^2)} \le \frac{1}{7}, \tag{2.5}$$

if A and b are sufficiently large.

Proof. We prove (2.5) by induction. For $k = 0$, from (2.2), it follows

$$\|S(\Phi_0^0)\|_{L^\infty(\mathbb{T}^2)} \le 2\pi\varepsilon N_0^{\beta+\alpha-1}$$

Thus, choosing A and b sufficiently large, we get

$$\|\frac{S(\Phi_0^0)}{N_1^{2\alpha}\varepsilon^2}\|_{L^\infty(\mathbb{T}^2)} \le 2\pi\varepsilon^{-1}N_0^{\beta+\alpha-1}N_1^{-2\alpha} \le 2\pi\varepsilon^{-1}A^{\beta+\alpha-1-2b\alpha} \le \frac{1}{7}. \tag{2.6}$$

Once again from (2.2), it yields that

$$\|\nabla^m \frac{S(\Phi_0^0)}{N_1^{2\alpha}\varepsilon^2}\|_{L^\infty(\mathbb{T}^2)} \lesssim N_0^m, \qquad \forall m \in \mathbb{N}. \tag{2.7}$$

Now, we suppose $\|\frac{S(\Phi_{k-1}^0)}{N_k^{2\alpha}\varepsilon^2}\|_{L^\infty(\mathbb{T}^2)} \le \frac{1}{7}$. Using the standard mollifier estimate or (2.7), we have

$$\|\nabla^m \frac{S(\Phi_{k-1}^0)}{N_k^{2\alpha}\varepsilon^2}\|_{L^\infty(\mathbb{T}^2)} \lesssim (N_{k-1}N_k)^{\frac{m}{2}}, \qquad \forall m \in \mathbb{N}. \tag{2.8}$$

By virtue of (2.3), (2.8) and the Leibnitz rule, we deduce

$$\|S(\Phi_k^0)\|_{L^\infty(\mathbb{T}^2)} \lesssim N_k^{-3+\beta+\alpha}(N_k^2 + N_{k-1}N_k) \lesssim N_k^{-1+\beta+\alpha},$$

which immediately implies

$$\|\frac{S(\Phi_k^0)}{N_{k+1}^{2\alpha}\varepsilon^2}\|_{L^\infty(\mathbb{T}^2)} \lesssim N_k^{-1+\beta+\alpha}N_{k+1}^{-2\alpha} \lesssim A^{b^k(\beta+\alpha-1-2b\alpha)} \le \frac{1}{7},$$

if A and b are sufficiently large. □

Now, we define the initial data in terms of the vector potentials introduced above.

Definition 2.4. *Define*

$$v_k^0 \triangleq \Delta\Phi_k^0.$$

and set the initial data by

$$V^0 \triangleq \sum_{k\in\mathbb{N}} v_k^0.$$

In the following, we establish the smallnes of the initial data V^0 in $B^{-\beta-\alpha}_{\infty,\infty}(\mathbb{T}^2)$.

Proposition 2.5. *With V^0 and $\{v_k^0\}_{k\in\mathbb{N}}$ as in Definition 2.4, there exists a universal constant $C > 0$, independent of ε, such that V^0 and $\{v_k^0\}_{k\in\mathbb{N}}$ are divergence-free and for any $k, m \in \mathbb{N}$ satisfy*

$$\|\nabla^m\Phi_k^0\|_{L^\infty(\mathbb{T}^2)} \lesssim N_k^{\beta+\alpha-2+m}, \tag{2.9}$$

$$\|\nabla^m v_k^0\|_{L^\infty(\mathbb{T}^2)} \lesssim N_k^{\beta+\alpha+m}, \tag{2.10}$$

and

$$\|V^0\|_{B^{-\beta-\alpha}_{\infty,\infty}(\mathbb{T}^2)} \le C\varepsilon.$$

Proof. The divergence-free property of V^0 and $\{v_k^0\}_{k\in\mathbb{N}}$ is easily derived from Definition 2.2. With Definition 2.2, Definition 2.4 and (2.8), one can use the Leibnitz rule to obtain (2.9), (2.10) and

$$\|\nabla^m a_{\xi,k}\|_{L^\infty(\mathbb{T}^2)} \lesssim N_{k-1}^m. \tag{2.11}$$

Before establishing the smallness of V^0 in $B^{-\beta-\alpha}_{\infty,\infty}(\mathbb{T}^2)$, for any $k\geq 1$, we decompose $v_k^0 = v_k^{0,p} + v_k^{0,e}$ where

$$\begin{aligned} v_k^{0,p} &= -N_k^{\beta+\alpha}\varepsilon\sqrt{2}(2\pi)^\beta \sum_{\xi\in\Lambda} a_{\xi,k} e^{2\pi i N_k \xi\cdot}\bar{\xi}, \\ v_k^{0,e} &= v_k^{0,e,1} + v_k^{0,e,2}, \end{aligned} \tag{2.12}$$

having defined

$$v_k^{0,e,1} = N_k^{-3+\beta+\alpha}\varepsilon\sqrt{2}(2\pi)^{\beta-3} i\Delta\nabla^\perp \left(\sum_{\xi\in\Lambda} a_{\xi,k}e^{2\pi i N_k\xi\cdot x}\right) + N_k^{\beta+\alpha}\varepsilon\sqrt{2}(2\pi)^\beta \sum_{\xi\in\Lambda} a_{\xi,k}e^{2\pi i N_k \xi\cdot}\bar{\xi}, \tag{2.13}$$

$$v_k^{0,e,2} = N_k^{-3+\beta+\alpha}\varepsilon\sqrt{2}(2\pi)^{\beta-3} i\Delta\nabla^\perp \left(\sum_{\xi\in\Lambda} a_{\xi,k}e^{2\pi i N_k\xi\cdot x}\right) * (\phi_k - \delta). \tag{2.14}$$

Next, we prove $\|\sum_{k\geq 1} v_k^{0,p}\|_{B^{-\beta-\alpha}_{\infty,\infty}(\mathbb{T}^2)} \lesssim \varepsilon$. Without loss of generality, fix $\xi\in\Lambda$. It suffices to show $\|\sum_{k\geq 1} N_k^{\beta+\alpha} a_{\xi,k} e^{2\pi i N_k\xi\cdot x}\|_{B^{-\beta-\alpha}_{\infty,\infty}(\mathbb{T}^2)} \lesssim 1$. Taking a fourier series of $\{a_{\xi,k}\}_{k\geq 1}$, we have

$$\begin{aligned} a_{\xi,k}(x) &= \sum_{l\in\mathbb{Z}^d} \hat{a}_{\xi,k}(l)e^{2\pi i l\cdot x}, \\ a_{\xi,k}(x)e^{2\pi i N_k\xi\cdot x} &= \sum_{l\in\mathbb{Z}^d} \hat{a}_{\xi,k}(l - N_k\xi)e^{2\pi i l\cdot x}. \end{aligned} \tag{2.15}$$

By (2.11) and Parseval's identity, we obtain

$$\|\nabla^m a_{\xi,k}\|_{L^2(\mathbb{T}^2)} = \left(\sum_{l\in\mathbb{Z}^d} |l|^{2m}|\hat{a}_{\xi,k}(l)|^2\right)^{\frac{1}{2}} \lesssim \|\nabla^m a_{\xi,k}\|_{L^\infty(\mathbb{T}^2)} \lesssim N_{k-1}^m, \qquad \forall m\in\mathbb{N},$$

which immediately implies

$$|l|^m|\hat{a}_{\xi,k}(l)| \lesssim N_{k-1}^m, \qquad \forall (l,m)\in\mathbb{Z}^d\times\mathbb{N}. \tag{2.16}$$

Recalling the definition of $B^{-\beta-\alpha}_{\infty,\infty}(\mathbb{T}^2)$ in Definition 1.4 and fixing $j\geq -1$, we decompose the following into three parts:

$$\begin{aligned} \Delta_j \sum_{k\in\mathbb{N}} N_k^{\beta+\alpha} a_{\xi,k}e^{2\pi i N_k\xi\cdot} &= \Delta_j\Big(\sum_{k:|2^j - N_k|\geq\frac{3}{4}N_k} + \sum_{k:|2^j-N_k|\leq\frac{3}{4}N_k}\Big) N_k^{\beta+\alpha} a_{\xi,k}e^{2\pi i N_k\xi\cdot} \\ &\triangleq I_1 + I_2. \end{aligned}$$

Consider I_1. If $j\geq 0$, by (2.15) and (2.16),

$$2^{-j(\beta+\alpha)}\|I_1\|_{L^\infty(\mathbb{T}^2)} \lesssim \sum_{k:|2^j-N_k|\geq\frac{3}{4}N_k} 2^{-j(\beta+\alpha)} N_k^{\beta+\alpha}\|\Delta_j(a_{\xi,k}e^{2\pi i N_k\xi\cdot})\|_{L^\infty(\mathbb{T}^2)}$$

$$
\begin{aligned}
&\lesssim \sum_{k:|2^j-N_k|\geq\frac{3}{4}N_k} 2^{-j(\beta+\alpha)}N_k^{\beta+\alpha} \sum_{\frac{3}{4}\cdot 2^j\leq|l|\leq\frac{8}{3}\cdot 2^j} |\hat{a}_{\xi,k}(l-N_k\xi)| \\
&\lesssim \sum_{k:|2^j-N_k|\geq\frac{3}{4}N_k} 2^{-j(\beta+\alpha)}N_k^{\beta+\alpha} \sum_{\frac{3}{4}\cdot 2^j\leq|l|\leq\frac{8}{3}\cdot 2^j} \frac{N_{k-1}^{\beta+\alpha+4}}{|l-N_k\xi|^{\beta+\alpha+4}} \\
&\lesssim \sum_{k\geq 1} N_k^{\beta+\alpha} \sum_{l\in\mathbb{Z}^2\setminus\{0\}} \frac{N_{k-1}^{\beta+\alpha+4}}{|l|^3 N_k^{\beta+\alpha+1}} \\
&\lesssim \sum_{k\geq 1} \frac{N_{k-1}^{\beta+\alpha+4}}{N_k} \sum_{l\in\mathbb{Z}^2\setminus\{0\}} \frac{1}{|l|^3} \\
&\lesssim 1, \qquad (2.17)
\end{aligned}
$$

where we use the fact that $|l-N_k\xi| \gtrsim \max\{|l|, N_k\}$ holds provided $|2^j - N_k| \geq \frac{3}{4}N_k$ and $\frac{3}{4}\cdot 2^j \leq |l| \leq \frac{8}{3}\cdot 2^j$. We also have $\sum_{k\geq 1}\frac{N_{k-1}^{\beta+\alpha+4}}{N_k} \lesssim 1$ if the parameter b is chosen sufficiently large. Similarly, if $j=-1$, we have

$$
\begin{aligned}
2^{\beta+\alpha}\|I_1\|_{L^\infty(\mathbb{T}^2)} &\lesssim \sum_{k:|\frac{1}{2}-N_k|\geq\frac{3}{4}N_k} 2^{\beta+\alpha}N_k^{\beta+\alpha} \sum_{|l|\leq\frac{4}{3}} |\hat{a}_{\xi,k}(l-N_k\xi)| \\
&\lesssim \sum_{k\geq 1} N_k^{\beta+\alpha} \sum_{|l|\leq\frac{4}{3}} \frac{N_{k-1}^{\beta+\alpha+1}}{|l-N_k\xi|^{\beta+\alpha+1}} \\
&\lesssim \sum_{k\geq 1} \frac{N_{k-1}^{\beta+\alpha+1}}{N_k} \\
&\lesssim 1, \qquad (2.18)
\end{aligned}
$$

where we use the estimate $|l-N_k\xi| \gtrsim N_k$ that is satisfied with $|\frac{1}{2}-N_k| \geq \frac{3}{4}N_k$ and $|l| \leq \frac{4}{3}$.

We now turn to the estimate of I_2. Since the bound $\|a_{\xi,k}e^{2\pi i N_k\xi\cdot}\|_{L^\infty(\mathbb{T}^2)} \lesssim 1$, we deduce

$$
\begin{aligned}
2^{-j(\beta+\alpha)}\|I_2\|_{L^\infty(\mathbb{T}^2)} &\lesssim \sum_{k:|2^j-N_k|\leq\frac{3}{4}N_k} 2^{-j(\beta+\alpha)}N_k^{\beta+\alpha}\|\Delta_j(a_{\xi,k}e^{2\pi i N_k\xi\cdot})\|_{L^\infty(\mathbb{T}^2)} \\
&\lesssim \sum_{k:|2^j-N_k|\leq\frac{3}{4}N_k} \|a_{\xi,k}e^{2\pi i N_k\xi\cdot}\|_{L^\infty(\mathbb{T}^2)} \\
&\lesssim \#\{K\leq 1 : |2^j - N_k| \leq \frac{3}{4}N_k\}.
\end{aligned}
$$

With a large enough choice of A and b, it holds

$$
\#\{K\leq 1 : |2^j - N_k| \leq \frac{3}{4}N_k\} \leq \log_{A^b}\frac{4\cdot 2^j}{\frac{4}{7}\cdot 2^j} + 1 \leq 2,
$$

which gives the bound

$$
2^{-j(\beta+\alpha)}\|I_2\|_{L^\infty(\mathbb{T}^2)} \lesssim 1. \qquad (2.19)
$$

Combining (2.17),(2.18) and (2.19) together, we get

$$
\|\sum_{k\geq 1} v_k^{0,p}\|_{B^{-\beta-\alpha}_{\infty,\infty}(\mathbb{T}^2)} \lesssim \varepsilon. \qquad (2.20)
$$

By a similar argument, we obtain

$$\|v_0^0\|_{B^{-\beta-\alpha}_{\infty,\infty}(\mathbb{T}^2)} \lesssim \varepsilon, \tag{2.21}$$

$$\|\sum_{k\geq 1} v_k^{0,e,1}\|_{B^{-\beta-\alpha}_{\infty,\infty}(\mathbb{T}^2)} \lesssim \varepsilon. \tag{2.22}$$

Since $v_k^{0,e,1}$ denotes the part where derivatives are not all applied to $e^{2\pi i N_k \xi\cdot x}$, the sum $\sum_{k\geq 1} v_k^{0,e,1}$ indeed enjoys smallness in stronger spaces, such as $B^{-\beta-\alpha+1-\lambda}_{\infty,\infty}$ for some $0<\lambda<1$.

We finish by estimating $\sum_{k\geq 1} v_k^{0,e,2}$. Recall that the length scale of the mollifier ϕ_k is $(N_{k+1}N_k)^{\frac{1}{2}}$. Applying the standard mollifier estimate and (2.11), we have

$$\begin{aligned}
\|\sum_{k\geq 1} v_k^{0,e,2}\|_{B^{-\beta-\alpha}_{\infty,\infty}(\mathbb{T}^2)} &\lesssim \sum_{k\geq 1} \varepsilon N_k^{-3+\beta+\alpha} \|\Delta\nabla^{\perp}\left(\sum_{\xi\in\Lambda} a_{\xi,k} e^{2\pi i N_k \xi\cdot}\right) * (\phi_k-\delta)\|_{L^\infty(\mathbb{T}^2)} \\
&\lesssim \sum_{k\geq 1} \varepsilon N_k^{-3+\beta+\alpha} \|\nabla^4 \left(\sum_{\xi\in\Lambda} a_{\xi,k} e^{2\pi i N_k \xi\cdot}\right)\|_{L^\infty(\mathbb{T}^2)} (N_{k+1}N_k)^{-\frac{1}{2}} \\
&\lesssim \sum_{k\geq 1} \varepsilon \frac{N_k^{\frac{1}{2}+\beta+\alpha}}{N_{k+1}^{\frac{1}{2}}} \\
&\lesssim \varepsilon,
\end{aligned} \tag{2.23}$$

where $\sum_{k\geq 1} \frac{N_k^{\frac{1}{2}+\beta+\alpha}}{N_{k+1}^{\frac{1}{2}}} \lesssim 1$ if the parameters A and b are chosen sufficiently large.

Combining (2.20)-(2.23) together, we conclude

$$\begin{aligned}
\|V^0\|_{B^{-\beta-\alpha}_{\infty,\infty}(\mathbb{T}^2)} &\leq \|\sum_{k\geq 1} v_k^{0,p}\|_{B^{-\beta-\alpha}_{\infty,\infty}(\mathbb{T}^2)} + \|v_0^0\|_{B^{-\beta-\alpha}_{\infty,\infty}(\mathbb{T}^2)} \\
&\quad + \|\sum_{k\geq 1} v_k^{0,e,1}\|_{B^{-\beta-\alpha}_{\infty,\infty}(\mathbb{T}^2)} + \|\sum_{k\geq 1} v_k^{0,e,2}\|_{B^{-\beta-\alpha}_{\infty,\infty}(\mathbb{T}^2)} \\
&\leq C\varepsilon.
\end{aligned}$$

□

2.2. Construction of the principal parts

Definition 2.6. *For any $k\in\mathbb{N}$, we define*

$$\begin{aligned}
v_k &\triangleq v_k^0 e^{-(2\pi N_k)^{2\beta} t}, \\
\bar{v}_k &\triangleq v_k^0 e^{-2\cdot(2\pi N_{k+1})^{2\beta} t}.
\end{aligned}$$

Then we define the principal parts of two solutions of (1.1)

$$v^{(1)} \triangleq \sum_{k\geq 0\, even} v_k + \sum_{k\geq 0\, odd} \bar{v}_k, \tag{2.24}$$

$$v^{(2)} \triangleq \sum_{k\geq 0\, odd} v_k + \sum_{k\geq 0\, even} \bar{v}_k. \tag{2.25}$$

Proposition 2.7. *Let $0 < p < \frac{2\beta}{\beta+\alpha}$. The two vector fields $v^{(1)}$ and $v^{(2)}$ are divergence-free, belong to $L^p([0,1];L^\infty(\mathbb{T}^2))$ and obey the following bounds*

$$\|\nabla^m v^{(i)}(t)\|_{L^\infty(\mathbb{T}^2)} \lesssim_m t^{-\frac{\beta+\alpha+m}{2\beta}} e^{-C_m N_0^{2\beta} t}, \tag{2.26}$$

for all $m \geq 0$, $t > 0$ and C_m is a constant depending only on m. Moreover, we have

$$\|v^{(i)}\|_{L^1([t_1,t_2],t^{-\frac{1}{2\beta}}\,dt;L^\infty(\mathbb{T}^2))} + \|v^{(i)}\|_{L^{\frac{2\beta}{2\beta-1}}([t_1,t_2];L^\infty(\mathbb{T}^2))} \lesssim 1 + (b\log A)^{-1}\log(\frac{t_2}{t_1}). \tag{2.27}$$

Proof. The divergence-free property of $v^{(i)}$ is inherited from v_k^0.

Using (2.10), we have

$$\|\nabla^m v^{(i)}(t)\|_{L^\infty(\mathbb{T}^2)} \lesssim \sum_{k\in\mathbb{N}} N_k^{\beta+\alpha+m} e^{-N_k^{2\beta} t}. \tag{2.28}$$

We analyze the series function on the right-hand side. As N_k grows rapidly, the series is dominated by its largest term. For $t \leq N_0^{-2\beta}$, the dominant contribution occurs near $N_k = t^{-\frac{1}{2\beta}}$, so the series is bounded by $O(t^{-\frac{\beta+\alpha+m}{2\beta}})$. For $t > N_0^{-2\beta}$, the summands decrease monotonically, yielding the upper bound $N_0^{\beta+\alpha+m} e^{-N_0^{2\beta} t} = O_m(t^{-\frac{\beta+\alpha+m}{2\beta}} e^{-C_m N_0^{2\beta} t})$. Therefore, it yields (2.26).

Applying (2.28) and using $0 < p < \frac{2\beta}{\beta+\alpha}$, one infers

$$\|v^{(i)}\|_{L^p([0,1];L^\infty(\mathbb{T}^2))} \lesssim \sum_{k\in\mathbb{N}} N_k^{\beta+\alpha} \|e^{-N_k^{2\beta} t}\|_{L^p([0,1]} \lesssim \sum_{k\in\mathbb{N}} N_k^{\beta+\alpha-\frac{2\beta}{p}} \lesssim 1.$$

To estimate $v^{(1)}$ and $v^{(2)}$ in $L^1([t_1,t_2],t^{-\frac{1}{2\beta}}\,dt;L^\infty(\mathbb{T}^2))$, it suffices to bound $\sum_{k\in\mathbb{N}} \|v_k(t)\|_{L^\infty(\mathbb{T}^2)}$ in $L^1([t_1,t_2],t^{-\frac{1}{2\beta}}\,dt)$, as $\|\bar{v}_k(t)\|_{L^\infty(\mathbb{T}^2)} \leq \|v_k(t)\|_{L^\infty(\mathbb{T}^2)}$. For $0 < \alpha < \beta - 1$, a better estimate holds. By viture of (2.10), we have

$$\|\sum_{k\in\mathbb{N}} v_k\|_{L^1([t_1,t_2],t^{-\frac{1}{2\beta}}\,dt;L^\infty(\mathbb{T}^2))} \lesssim \sum_{k\in\mathbb{N}} N_k^{\beta+\alpha} N_k^{-2\beta} t_1^{-\frac{1}{2\beta}} e^{-N_k^{2\beta} t_1} \lesssim t_1^{\frac{\beta-1-\alpha}{2\beta}} \lesssim 1, \tag{2.29}$$

where we use $0 < \alpha < \beta - 1$ and $t_1 < 1$. For $\alpha = \beta - 1$, we split the intergrand into two parts:

$$\begin{aligned}
\|\sum_{k\in\mathbb{N}} v_k\|_{L^1([t_1,t_2],t^{-\frac{1}{2\beta}}\,dt;L^\infty(\mathbb{T}^2))} &\lesssim \sum_{k\in\mathbb{N}} \int_{t_1}^{t_2} \Big(\|v_k(t)\|_{L^\infty(\mathbb{T}^2)} \mathbf{1}_{t\leq N_k^{-2\beta}} + \|v_k(t)\|_{L^\infty(\mathbb{T}^2)} \mathbf{1}_{t\geq N_k^{-2\beta}}\Big) t^{-\frac{1}{2\beta}}\,dt \\
&\lesssim \sum_{k:N_k\leq t_1^{-1/2\beta}} \int_{t_1}^{N_k^{-2\beta}\wedge t_2} N_k^{2\beta-1} t^{-\frac{1}{2\beta}}\,dt \\
&\quad + \sum_{k:N_k\geq t_2^{-1/2\beta}} \int_{N_k^{-2\beta}\vee t_1}^{t_2} N_k^{2\beta-1} t^{-\frac{1}{2\beta}} e^{-N_k^{2\beta} t}\,dt \\
&\triangleq T_1 + T_2.
\end{aligned} \tag{2.30}$$

For T_1, direct computation yields

$$\begin{aligned}
T_1 &\lesssim \sum_{k:N_k\leq t_1^{-1/2\beta}} N_k^{2\beta-1} (N_k^{-2\beta} \wedge t_2)^{\frac{2\beta-1}{2\beta}} \\
&\lesssim \sum_{k:N_k\leq t_2^{-1/2\beta}} N_k^{2\beta-1} t_2^{\frac{2\beta-1}{2\beta}} + \sum_{k:t_2^{-1/2\beta}\leq N_k\leq t_1^{-1/2\beta}} 1
\end{aligned}$$

$$
\begin{aligned}
&\lesssim 1 + \#\{k \in \mathbb{N} : N_k \in [t_2^{-1/2\beta}, t_1^{-1/2\beta}]\} \\
&\lesssim 1 + (b \log A)^{-1} \log(\frac{t_2}{t_1}),
\end{aligned} \tag{2.31}
$$

where we employ the estimate

$$
\#\{k \in \mathbb{N} : N_k \in [t_2^{-1/2\beta}, t_1^{-1/2\beta}]\} \leq 1 + \log_{A^b} \frac{t_1^{-1/2\beta}}{t_2^{-1/2\beta}} \lesssim 1 + (b \log A)^{-1} \log(\frac{t_2}{t_1}).
$$

For T_2, with a similar argument, we have

$$
\begin{aligned}
T_2 &\lesssim \sum_{k: N_k \geq t_2^{-1/2\beta}} N_k^{-1} (N_k^{-2\beta} \vee t_1)^{-\frac{1}{2\beta}} e^{-N_k^{2\beta}(N_k^{-2\beta} \vee t_1)} \\
&\lesssim \sum_{k: t_2^{-1/2\beta} \leq N_k \leq t_1^{-1/2\beta}} 1 + \sum_{k: N_k \geq t_1^{-1/2\beta}} N_k^{-1} t_1^{-\frac{1}{2\beta}} \\
&\lesssim 1 + (b \log A)^{-1} \log(\frac{t_2}{t_1}).
\end{aligned} \tag{2.32}
$$

Combining (2.29)-(2.32) together, we arrive at (2.27) while the estimates in $L^{\frac{2\beta}{2\beta-1}}([t_1, t_2]; L^\infty(\mathbb{T}^2))$ are directly derived from the bound on $L^1([t_1, t_2], t^{-\frac{1}{2\beta}} dt; L^\infty(\mathbb{T}^2))$ and (2.26) under the condition $0 < \alpha \leq \beta - 1$. $\square$

2.3. Estimates on the residual

For $i \in \{1, 2\}$, we define the residual $F^{(i)} \in C^\infty((0, 1] \times \mathbb{T}^2; S^{2\times 2})$ in divergence form, satisfying

$$
\mathbb{P} \operatorname{div} F^{(i)} = \partial_t v^{(i)} + (-\Delta)^\beta v^{(i)} + \mathbb{P} \operatorname{div}(v^{(i)} \otimes v^{(i)}). \tag{2.33}
$$

Let $\gamma \leq \frac{2-\beta-\alpha}{2}$ and $0 < \kappa \leq \frac{\gamma}{2}$. We consider the Banach space Y^α equipped with a subcritical norm

$$
\|F\|_{Y^\alpha} \triangleq \sup_{t \in (0,1]} t^{\frac{3\beta+\alpha-2+\gamma}{2\beta}} \|F(t)\|_{L^\infty(\mathbb{T}^2)} + t^{\frac{3\beta+\alpha-1+\gamma}{2\beta}} \|\nabla F(t)\|_{C^\kappa(\mathbb{T}^2)}.
$$

Proposition 2.8. *For any $\eta > 0$, there exists $F^{(1)}, F^{(2)} \in C^\infty((0, 1] \times \mathbb{T}^2; S^{2\times 2})$ satisfying*

$$
\|F^{(i)}\|_{Y^\alpha} \leq \eta,
$$

if A and b are sufficiently large.

Proof. We proceed to verify that there exist $F^{(1)}$ that satisfies the desired properties. The argument for $F^{(2)}$ is analogous and will be omitted. Recalling (2.12), (2.13) and (2.14), for any $k \geq 1$, we decompose $v_k = v_k^p + v_k^e$ where

$$
\begin{aligned}
v_k^p &\triangleq v_k^{0,p} e^{-(2\pi N_k)^{2\beta} t}, \\
v_k^e &\triangleq (v_k^{0,e,1} + v_k^{0,e,2}) e^{-(2\pi N_k)^{2\beta} t}.
\end{aligned} \tag{2.34}
$$

Using (2.11), the standard mollifier estimate and the Leibnitz rule, we have

$$
\|\nabla^m v_k^p\|_{L^\infty(\mathbb{T}^2)} \lesssim N_k^{\beta+\alpha+m} e^{-N_k^{2\beta} t}, \tag{2.35}
$$

$$\|\nabla^m v_k^e\|_{L^\infty(\mathbb{T}^2)} \lesssim \left(\frac{N_{k-1}}{N_k} + \frac{N_k^{\frac{1}{2}}}{N_{k+1}^{\frac{1}{2}}}\right) N_k^{\beta+\alpha+m} e^{-N_k^{2\beta} t} \tag{2.36}$$

$$\|\nabla^m \bar{v}_k\|_{L^\infty(\mathbb{T}^2)} \lesssim N_k^{\beta+\alpha+m} e^{-N_{k+1}^{2\beta} t}, \tag{2.37}$$

for all $k \geq 1$ and $m \in \mathbb{N}$. We denote

$$v^p \triangleq \sum_{k\geq 0, even} v_k^p,$$
$$\widetilde{v} \triangleq \sum_{k\geq 1\, odd} \bar{v}_k + \sum_{k\geq 2\, even} v_k^e.$$

Due to (2.35), we know that v^p admits the same estimate as $v^{(1)}$,

$$\|\nabla^m v^p(t)\|_{L^\infty(\mathbb{T}^2)} \lesssim t^{\frac{\beta+\alpha+m}{2\beta}}. \tag{2.38}$$

(2.36) and (2.37) imply

$$\begin{aligned}
\|\nabla^m \widetilde{v}(t)\|_{L^\infty(\mathbb{T}^2)} &\lesssim \sum_{k\geq 1} \left(\frac{N_{k-1}}{N_k} + \frac{N_k^{\frac{1}{2}}}{N_{k+1}^{\frac{1}{2}}}\right) N_k^{\beta+\alpha+m} e^{-N_k^{2\beta} t} + N_k^{\beta+\alpha+m} e^{-N_{k+1}^{2\beta} t} \\
&\lesssim \sum_{k\in\mathbb{N}} \left(N_k^{1/b-1} + N_k^{1/2-b/2} + N_k^{(\beta+\alpha+m)(\frac{1}{b}-1)}\right) N_k^{\beta+\alpha+m} e^{-N_k^{2\beta} t} \\
&\lesssim \sum_{k\in\mathbb{N}} N_k^{1/b-1} N_k^{\beta+\alpha+m} e^{-N_k^{2\beta} t}
\end{aligned}$$

Therefore, we deduce

$$\sup_{t\in(0,1]} t^{\frac{\beta+\alpha-1+\gamma/2}{2\beta}} \|\widetilde{v}(t)\|_{L^\infty(\mathbb{T}^2)} + t^{\frac{\beta+\alpha+\gamma/2+\kappa}{2\beta}} \|\nabla \widetilde{v}(t)\|_{C^\kappa(\mathbb{T}^2)} \lesssim \sup_{k\geq 1} N_k^{1/b-\frac{\gamma}{2}} \lesssim \eta, \tag{2.39}$$

where we first take b sufficiently large such that $1/b - \frac{\gamma}{2} < 0$ and then choose A large enough.

Recall the differential operator S in Definition 2.1. For any $k \in \mathbb{N}$, we define the tensor fields

$$R_k \triangleq S(\Phi_k^0) e^{-(2\pi N_k)^{2\beta} t},$$
$$\bar{R}_k \triangleq S(\Phi_k^0) e^{-2\cdot(2\pi N_{k+1})^{2\beta} t}.$$

Thus, by (2.1) and Definition 2.6, it holds

$$\mathbb{P} \operatorname{div} R_k = v_k \quad \text{and} \quad \mathbb{P} \operatorname{div} \bar{R}_k = \bar{v}_k.$$

Let us define the residual term $F^{(1)}$ as follows,

$$F^{(1)} \triangleq F_1^{(1)} + F_2^{(1)} + F_3^{(1)} + F_4^{(1)},$$

where

$$F_1^{(1)} = -\sum_{k\geq 0\, even} (\partial_t + (-\Delta)^\beta) R_k,$$
$$F_2^{(1)} = -\sum_{k\geq 1\, odd} (-\Delta)^\beta \bar{R}_k,$$

$$F_3^{(1)} = -\sum_{k\geq 1\, odd} \mathcal{R}\operatorname{div}(\partial_t \bar{R}_k + v_{k+1}^p \otimes v_{k+1}^p) - \mathcal{R}\nabla p_{k+1},$$
$$F_4^{(1)} = -v^{(1)} \otimes v^{(1)} + \sum_{k\geq 0\, even} v_k^p \otimes v_k^p.$$

Here $\{\nabla p_k\}_{k\geq 1}$ denote some pressure terms, which will be defined later. As $\mathbb{P}\operatorname{div}\mathcal{R}\nabla p_k = \mathbb{P}\nabla p_k = 0$, the term $F_3^{(1)}$ is well-defined and we can easily verify (2.33).

Estimate of $F_1^{(1)}$. By (2.3) and the fact that $\{e^{2\pi i N_k x} e^{-(2\pi N_k)^{2\beta} t}\}_{k\geq 1}$ and v_0 are solutions to the heat equation $\left(\partial_t + (-\Delta)^\beta\right) u = 0$, we derive

$$F_1^{(1)}(t,x) = -\sum_{k\geq 1\, even}\sum_{\xi\in\Lambda} N_k^{-3+\beta+\alpha} \varepsilon\sqrt{2}(2\pi)^{\beta-3} i S\left(\nabla^\perp[(-\Delta)^\beta, a_{\xi,k}(x)] e^{2\pi i N_k \xi\cdot x}\right) * \phi_k e^{-(2\pi N_k)^{2\beta} t}.$$

Since S is a differential operator of degree 1. Thanks to (2.11), (B.2) and the standard mollifier estimates, one infers

$$\begin{aligned}
\|\nabla^m F_1^{(1)}(t)\|_{L^\infty(\mathbb{T}^2)} &\lesssim \sum_{k\geq 1}\sum_{\xi\in\Lambda} N_k^{-3+\beta+\alpha} \|\nabla^{2+m}[(-\Delta)^\beta, a_{\xi,k}] e^{2\pi i N_k \xi\cdot}\|_{L^\infty(\mathbb{T}^2)} e^{-(2\pi N_k)^{2\beta} t} \\
&\lesssim \sum_{k\geq 1} N_k^{-3+\beta+\alpha} N_k^{2+m}\left(N_k^{2\beta-1} N_{k-1}^4 + N_{k-1}^{2\beta+3}\right) e^{-N_k^{2\beta} t} \\
&\lesssim \sum_{k\in\mathbb{N}} N_k^{3\beta+\alpha-2+\frac{4}{b}+m} e^{-N_k^{2\beta} t}.
\end{aligned}$$

Therefore, using $\kappa \leq \frac{\gamma}{2}$ and choosing A, b sufficiently large, we obtain

$$\|F_1^{(1)}\|_{Y^\alpha} \lesssim \sup_k N_k^{\frac{4}{b}+\kappa-\gamma} \lesssim \eta. \tag{2.40}$$

Estimate of $F_2^{(1)}$. We estimate directly

$$\begin{aligned}
\|\nabla^m F_2^{(1)}(t)\|_{L^\infty(\mathbb{T}^2)} &\lesssim \sum_{k\geq 1\, odd} N_k^{3\beta+\alpha-1+m} e^{-N_{k+1}^{2\beta} t} \\
&\lesssim \sum_{k\in\mathbb{N}} N_k^{\frac{3\beta+\alpha-1+m}{b}} e^{-N_k^{2\beta} t}.
\end{aligned}$$

Hence, we easily deduce

$$\|F_2^{(1)}\|_{Y^\alpha} \lesssim \eta, \tag{2.41}$$

if A and b are large enough.

Estimate of $F_3^{(1)}$. Recalling (2.12) and (2.34), we have

$$\operatorname{div}(v_k^p \otimes v_k^p) = \operatorname{div}\left(\sum_{\xi_1,\xi_2\in\Lambda} a_{\xi_1,k} a_{\xi_2,k} e^{2\pi i N_k(\xi_1+\xi_2)\cdot x} \bar{\xi}_1 \otimes \bar{\xi}_2\right) 2\cdot(2\pi)^{2\beta} N_k^{2\beta+2\alpha} \varepsilon^2 e^{-2\cdot(2\pi N_k)^{2\beta} t}. \tag{2.42}$$

Applying Lemma A.4 and the product rule, we have

$$\operatorname{div}\left(\sum_{\xi_1,\xi_2\in\Lambda} a_{\xi_1,k} a_{\xi_2,k} e^{2\pi i N_k(\xi_1+\xi_2)\cdot x} \bar{\xi}_1 \otimes \bar{\xi}_2\right) \tag{2.43}$$

$$
\begin{aligned}
&= \sum_{\xi\in\Lambda} \operatorname{div}(a_{\xi,k}^2 \bar{\xi}\otimes\bar{\xi}) + \sum_{\xi_1\neq\xi_2\in\Lambda} \operatorname{div}(a_{\xi_1,k}a_{\xi_2,k}\bar{\xi}_1\otimes\bar{\xi}_2)e^{2\pi i N_k(\xi_1+\xi_2)x} \\
&\quad - \frac{1}{2}\nabla(|v_k^{0,p}|^2 - |\sum_{\xi\in\Lambda} a_{\xi,k}e^{2\pi i N_k\xi\cdot x}|^2) + \frac{1}{2}\sum_{\xi_1\neq\xi_2\in\Lambda}\nabla(a_{\xi_1,k}a_{\xi_2,k})(\bar{\xi}_1\cdot\bar{\xi}_2-1)e^{2\pi i N_k(\xi_1-\xi_2)x}. \qquad (2.44)
\end{aligned}
$$

Moreover, using (2.4) and Lemma A.1, we have

$$
\sum_{\xi\in\Lambda}\operatorname{div}(a_{\xi,k}^2\bar{\xi}\otimes\bar{\xi}) = \operatorname{div}\Big(\frac{S(\Phi_{k-1}^0)}{N_k^{2\alpha}\varepsilon^2} + \mathrm{Id}\Big) = \frac{v_{k-1}^0}{N_k^{2\alpha}\varepsilon^2}.
$$

Therefore, we have

$$
\operatorname{div}\left(\partial_t\bar{R}_k + \sum_{\xi\in\Lambda} a_{\xi,k+1}^2\bar{\xi}\otimes\bar{\xi}\cdot 2(2\pi)^{2\beta}N_{k+1}^{2\beta+2\alpha}\varepsilon^2 e^{-2\cdot(2\pi N_{k+1})^{2\beta}t}\right) = 0. \qquad (2.45)
$$

Combining (2.42)-(2.45) together, we simplify $F_3^{(1)}$ as follows

$$
\begin{aligned}
F_3^{(1)} &= -\sum_{k\geq 1\, odd}\mathcal{R}\operatorname{div}(\partial_t\bar{R}_k + v_{k+1}^p\otimes v_{k+1}^p) - \mathcal{R}\nabla p_{k+1} \\
&= -\Big(\sum_{k\geq 1\, odd}\mathcal{R}\sum_{\xi_1\neq-\xi_2\in\Lambda}\operatorname{div}(a_{\xi_1,k}a_{\xi_2,k}\bar{\xi}_1\otimes\bar{\xi}_2)e^{2\pi i N_k(\xi_1+\xi_2)x} \\
&\quad + \sum_{k\geq 1\, odd}\mathcal{R}\sum_{\xi_1\neq\xi_2\in\Lambda}\frac{1}{2}\nabla(a_{\xi_1,k}a_{\xi_2,k})(\bar{\xi}_1\cdot\bar{\xi}_2-1)e^{2\pi i N_k(\xi_1-\xi_2)x}\Big) \\
&\quad \cdot 2\cdot(2\pi)^{2\beta}N_k^{2\beta+2\alpha}\varepsilon^2 e^{-2\cdot(2\pi N_k)^{2\beta}t},
\end{aligned}
$$

where $p_k = \frac{1}{2}(|v_k^{0,p}|^2 - |\sum_{\xi\in\Lambda}a_{\xi,k}e^{2\pi i N_k\xi\cdot x}|^2)$.

Suppose $\{b_k\}_{k\geq 1}$ are smooth functions with $\|\nabla^m b_k\|_{L^\infty}\lesssim N_{k-1}^m$ for any $m\in\mathbb{N}$ and Let $\xi\in\mathbb{Z}^d\backslash\{0\}$. Without loss of generality, it suffices to show that

$$
\|\sum_{k\geq 1}\mathcal{R}(\nabla b_k e^{2\pi i N_k\xi\cdot x})N_k^{2\beta+2\alpha}e^{-(2\pi N_k)^{2\beta}t}\|_{Y^\alpha}\lesssim\eta.
$$

Using (A.2) by choosing m large enough, we have, for any $k\geq 1$,

$$
\|\mathcal{R}(\nabla b_k e^{2\pi i N_k\xi\cdot x})\|_{C^\mu(\mathbb{T})^2}\lesssim N_k^{-1+\mu}N_{k-1} + N_k^{-m+\mu}N_{k-1}^{m+1} + N_k^{-m}N_{k-1}^{m+\mu+1}\lesssim N_k^{-1+\mu}N_{k-1}.
$$

Thus, setting $\mu=\frac{\gamma}{2}$, choosing A,b sufficiently large, and using $\alpha\leq\beta-1$, we arrive at

$$
\begin{aligned}
\|\sum_{k\geq 1}\mathcal{R}(\nabla b_k e^{2\pi i N_k\xi\cdot x})N_k^{2\beta+2\alpha}e^{-(2\pi N_k)^{2\beta}t}\|_{C^\mu(\mathbb{T})^2} &\lesssim \sum_{k\geq 1}N_k^{2\beta+2\alpha-1+\mu}N_{k-1}e^{-(N_k)^{2\beta}t} \\
&\lesssim \sup_k N_k^{-\beta+\alpha+1-\frac{\gamma}{2}+\frac{1}{b}}t^{-\frac{3\beta+\alpha-2+\gamma}{2\beta}} \\
&\lesssim \eta t^{-\frac{3\beta+\alpha-2+\gamma}{2\beta}}. \qquad (2.46)
\end{aligned}
$$

On the other hand, applying the fact that $\nabla\mathcal{R}$ is bounded from $C^\kappa(\mathbb{T}^2)$ to $C^\kappa(\mathbb{T}^2)$, one infers

$$
\|\sum_{k\geq 1}\nabla\mathcal{R}(\nabla b_k e^{2\pi i N_k\xi\cdot x})N_k^{2\beta+2\alpha}e^{-(2\pi N_k)^{2\beta}t}\|_{C^\kappa(\mathbb{T})^2}\lesssim\sum_{k\geq 1}N_{k-1}N_k^{2\beta+2\alpha+\kappa}e^{-(N_k)^{2\beta}t},
$$

$$
\begin{aligned}
&\lesssim \sum_{k\geq 1} N_k^{2\beta+2\alpha+\kappa+\frac{1}{b}} e^{-(N_k)^{2\beta}t} \\
&\lesssim \sup_k N_k^{-\beta+\alpha+1+\kappa-\gamma+\frac{1}{b}} t^{-\frac{3\beta+\alpha-1+\gamma}{2\beta}} \\
&\lesssim \eta t^{-\frac{3\beta+\alpha-1+\gamma}{2\beta}}, \qquad (2.47)
\end{aligned}
$$

where we use $\kappa \leq \frac{\gamma}{2}$ and choose A, b sufficiently large again. Combing (2.46) and (2.47) together, we get the desired result and thus we have

$$\|F_3^{(1)}\|_{Y^\alpha} \lesssim \eta, \qquad (2.48)$$

Estimate of $F_4^{(1)}$. We decompose $F_4^{(1)}$ as

$$F_4^{(1)} = -\sum_{k\neq j\geq 0\, even} v_k^p \otimes v_j^p - v^{(1)} \otimes \widetilde{v} - \widetilde{v} \otimes v^p.$$

Applying (2.35), for any $m \in \mathbb{N}$, we obtain

$$
\begin{aligned}
\|\nabla^m \sum_{k\neq j} v_k^p \otimes v_j^p\|_{L^\infty(\mathbb{T}^2)} &\lesssim \sum_{k\neq j} \|\nabla^m (v_k^p \otimes v_j^p)\|_{L^\infty(\mathbb{T}^2)} \\
&\lesssim \sum_{k\neq j} (N_k + N_j)^m N_k^{\beta+\alpha} N_j^{\beta+\alpha} e^{-N_k^{2\beta}t - N_j^{2\beta}t} \\
&\lesssim \sum_{k\geq 1} N_k^{\beta+\alpha+m} e^{-N_k^{2\beta}t} \sum_{j<k} N_j^{\beta+\alpha} \\
&\lesssim \sum_{k\geq 1} N_k^{(\beta+\alpha)(\frac{1}{b}+1)+m} e^{-N_k^{2\beta}t}.
\end{aligned}
$$

Therefore, it follows

$$
\begin{aligned}
\|\sum_{k\neq j} v_k^p \otimes v_j^p\|_{Y^\alpha} &\lesssim \sup_{k\geq 1} N_k^{(\beta+\alpha)(\frac{1}{b}+1)+1+\kappa-(3\beta+\alpha-1+\gamma)} \\
&\lesssim \sup_{k\geq 1} N_k^{2-2\beta+(\beta+\alpha)\frac{1}{b}+\kappa-\gamma} \\
&\lesssim \eta, \qquad (2.49)
\end{aligned}
$$

where we have used the fact that $\beta > 1$ and $\kappa \leq \frac{\gamma}{2}$, and chosen A and b large enough. By viture of (2.10), (2.38) and (2.39), we deduce

$$\|v^{(1)} \otimes \widetilde{v} + \widetilde{v} \otimes v^p\|_{Y^\alpha} \lesssim \sup_{t\in(0,1]} t^{\frac{\gamma/2-\kappa+\beta-1-\alpha}{2\beta}} \eta \lesssim \eta, \qquad (2.50)$$

where we have used the fact that $\alpha \leq \beta - 1$ and $\kappa \leq \frac{\gamma}{2}$. Thus, (2.49) and (2.50) yield

$$\|F_4^{(1)}\|_{Y^\alpha} \lesssim \eta. \qquad (2.51)$$

Combining (2.40), (2.41), (2.48) and (2.51), and taking a new η small enough to absorb all constants, we conclude that

$$\|F^{(1)}\|_{Y^\alpha} \leq \eta.$$

□

3 Construction of the perturbation

$v^{(1)}$ and $v^{(2)}$ we constructed in the above section are almost solutions of (1.1). To show this, we prove that the residual terms are small in a subcritical norm. Consequently, for $i \in \{1,2\}$, we can add a perturbation $\omega^{(i)}$ to $v^{(i)}$ to obtain an exact solution.

3.1. Semigroup of the linearization around the principal part

Let $\tau = \frac{2-\alpha-\beta-\gamma}{2\beta}$, Note that $\tau > 0$ as $\gamma < \frac{2-\alpha-\beta}{2}$. we rewrite the norm of Y^α

$$\|a\|_{Y^\alpha} := \sup_{t\in(0,1]} (t^{1-\tau}\|a\|_{L^\infty} + t^{1+\frac{1}{2\beta}-\tau}\|\nabla a\|_{C^\kappa}).$$

For $i \in \{1,2\}$, $f \in Y^\alpha$, and $0 < t' \le t \le 1$, we define the semigroup $P^{(i)}(t,t')f$ as the solution of

$$\begin{cases} (\partial_t + (-\Delta)^\beta)P^{(i)}(t,t')f + \mathbb{P}\,\mathrm{div}\left(P^{(i)}(t,t')f \otimes v^{(i)}(t) + v^{(i)}(t) \otimes P^{(i)}(t,t')f\right) = 0, \\ P^{(i)}(t',t')f = \mathbb{P}\,\mathrm{div}\ f(t'). \end{cases}$$

Because $v^{(i)}$ is smooth with uniform estimates on $[t',1] \times \mathbb{T}^2$, we then have $P^{(i)}(t,t')$ is well-defined and smooth. The following proposition shows that $P^{(i)}(t,t')$ behaves comparably to $e^{(t'-t)(-\Delta)^\beta}\mathbb{P}\mathrm{div}$ up to a small loss of the form $\left(\frac{t}{t'}\right)^\epsilon$, where $\epsilon > 0$ can be chosen arbitrarily small. We note that this loss arises because $v^{(i)}$ does not belong to $L^{\frac{2\beta}{2\beta-1}}([0,1];L^\infty)$ or $L^1([0,1],t^{-\frac{1}{2\beta}}dt;L^\infty)$.

Proposition 3.1. *For all $i \in 1,2$, $f \in Y^\alpha$, and $0 < t' \le t \le 1$, we have*

$$\|P^{(i)}(t,t')f\|_{L^\infty(\mathbb{T}^5)} + (t-t')^{\frac{1}{2\beta}}t^{\frac{\kappa}{2\beta}}\|\nabla P^{(i)}(t,t')f\|_{C^\kappa(\mathbb{T}^5)} \lesssim (t')^{-1+\tau-\epsilon}t^{-\frac{1}{2\beta}+\epsilon}\|f\|_{Y^\alpha}.$$

Proof. Through this proof, we suppress the dependence on i of $v^{(i)}$ and $P^{(i)}(t,t')$ with $i \in \{1,2\}$. For a fixed $t' \in (0,1)$, applying the Duhamel formula, we rewrite $P(t,t')f$ and split it into three parts:

$$\begin{aligned} P(t,t')f &= e^{(t'-t)(-\Delta)^\beta}\mathbb{P}\mathrm{div}f(t') - \int_{t'}^t e^{(s-t)(-\Delta)^\beta}\mathbb{P}\mathrm{div}\Big(P^{(i)}(t,t')f \otimes v^{(i)}(t) + v^{(i)}(t) \otimes P^{(i)}(t,t')f)ds \\ &:= I + II + III. \end{aligned}$$

where I is the linear heat operator terms, II is the integral parts over over $[t', t' \vee (\frac{t}{2})]$, and III is the integral parts over $[t' \vee (\frac{t}{2}), t]$.

We first focus on I. By applying Lemma C.1 to either $e^{(t-t')\Delta}\mathbb{P}$ or $e^{(t-t')\Delta}\mathbb{P}\mathrm{div}$, we choose the more favorable upper bound and have

$$\begin{aligned} \|I\|_{L^\infty} &\lesssim \|\nabla f(t')\|_{C^\kappa} \wedge (t-t')^{-\frac{1}{2\beta}}\|f(t')\|_{L^\infty} \\ &\lesssim (t')^{-1-\frac{1}{2\beta}+\tau} \wedge ((t-t')^{-\frac{1}{2\beta}}(t')^{-1+\tau})\|f\|_{Y^\alpha} \\ &\lesssim (t')^{-1+\tau}(t' \vee (t-t'))^{-\frac{1}{2\beta}}\|f\|_{Y^\alpha} \\ &\lesssim (t')^{-1+\tau}t^{-\frac{1}{2\beta}}\|f\|_{Y^\alpha}. \end{aligned} \tag{3.1}$$

Next, we consider II. Together with the fact that the integral vanishes identically unless $t > 2t'$, we obtain

$$\|II\|_{L^\infty} \lesssim \int_{t'}^{t'\vee(\frac{t}{2})} (t-s)^{-\frac{1}{2\beta}}\|v(s)\|_{L^\infty}\|P(s,t')f\|_{L^\infty}ds$$

$$\lesssim t^{-\frac{1}{2\beta}} \int_{t'}^{t'\vee(\frac{t}{2})} (t-s)^{-\frac{1}{2\beta}} \|v(s)\|_{L^\infty} \|P(s,t')f\|_{L^\infty} s^{\frac{1}{2\beta}} ds. \tag{3.2}$$

For III, by Lemma C.1 and the fact that $(\frac{t}{s})^{\frac{1}{2}} \le \sqrt{2}$ on the interval $s \in [t' \vee (\frac{t}{2}), t]$, it is easy to see that

$$\begin{aligned}\|III\|_{L^\infty} &\lesssim \int_{t'\vee(\frac{t}{2})}^{t} (t-s)^{-\frac{1}{2\beta}} \|v(s)\|_{L^\infty} \|P(s,t')f\|_{L^\infty} ds \\ &\lesssim t^{-\frac{1}{2\beta}} \int_{t'\vee(\frac{t}{2})}^{t} (t-s)^{-\frac{1}{2\beta}} \|v(s)\|_{L^\infty} \|P(s,t')f\|_{L^\infty} s^{\frac{1}{2\beta}} ds. \end{aligned} \tag{3.3}$$

Letting $h(t) := t^{\frac{1}{2\beta}} \|P(s,t')f\|_{L^\infty}$ and summing the estimates from (3.1) to (3.3), we get that

$$h(t) \lesssim (t')^{-1+\tau} \|f\|_Y + \int_{t'}^{t} (s^{-\frac{1}{2\beta}} + (t-s)^{-\frac{1}{2\beta}}) \|v(s)\|_{L^\infty} h(s) ds.$$

Now together with (2.26), (2.27) and Lemma C.2 with $p = 3$, $g_1(s) = s^{-\frac{1}{2\beta}} \|v(s)\|_{L^\infty}$ and $g_2(s) = \|v(s)\|_{L^\infty}$, we then have

$$\begin{aligned}h(t) &\lesssim (t')^{-1+\tau} \|f\|_{Y^\alpha} \exp(O(\|v\|_{L^1([t',t],t^{-\frac{1}{2\beta}}dt;L^\infty)} + \|s^{\frac{1}{2\beta}} v\|_{L^\infty_{t,x}([t',t])} \|v\|_{L^2([t',t];L^\infty)})) \\ &\lesssim (t')^{-1+\tau} \|f\|_{Y^\alpha} \exp(O(1 + (\log A)^{-1} \log(\frac{t}{t'}))).\end{aligned}$$

Choosing A sufficiently large depending on ϵ, the power becomes smaller than $\frac{\epsilon}{2}$ and we conclude that

$$\|P(t,t')f\|_{L^\infty} \lesssim t^{-\frac{1}{2\beta}} (t')^{-1+\tau} (\frac{t}{t'})^{\frac{\epsilon}{2}} \|f\|_{Y^\alpha}. \tag{3.4}$$

Now consider $C^{1,k}$ norm. I is unchanged while II split the integral into $[t', \frac{(t+t')}{2}]$, and III split the integral into $[\frac{(t+t')}{2}, t]$. About ∇I, we similarly have

$$\begin{aligned}\|\nabla I\|_{C^\kappa} &\lesssim ((t-t')^{-\frac{1}{2\beta}} \|\nabla f(t')\|_{C^\kappa}) \wedge ((t-t')^{-\frac{2+\kappa}{2\beta}} \|f(t')\|_{L^\infty}) \\ &\lesssim ((t-t')^{-\frac{1}{2\beta}} (t')^{-1-\frac{1}{2\beta}+\tau}) \wedge ((t-t')^{-\frac{2+\kappa}{2\beta}} (t')^{-1+\tau}) \|f\|_{Y^\alpha} \\ &\lesssim (t-t')^{-\frac{1}{2\beta}} (t')^{-1+\tau} t^{-\frac{1+\kappa}{2\beta}} \|f\|_{Y^\alpha}.\end{aligned} \tag{3.5}$$

For ∇II, combining with Lemma C.1, (2.26) and (3.4), we get

$$\begin{aligned}\|\nabla II\|_{C^\kappa} &\lesssim \int_{t'}^{\frac{(t+t')}{2}} (t-s)^{-\frac{2+\kappa}{2\beta}} \|v(s)\|_{L^\infty} \|P(s,t')f\|_{L^\infty} ds \\ &\lesssim (t')^{-1+\tau-\frac{\epsilon}{2}} \int_{t'}^{\frac{(t+t')}{2}} (t-s)^{-\frac{2+\kappa}{2\beta}} s^{-\frac{\beta+\alpha+1}{2\beta}+\frac{\epsilon}{2}} \|f\|_{Y^\alpha} ds \\ &\lesssim (t-t')^{1-\frac{2+\kappa}{2\beta}} (t')^{-1+\tau-\frac{\epsilon}{2}} t^{-\frac{\beta+\alpha+1}{2\beta}+\frac{\epsilon}{2}} \|f\|_{Y^\alpha}.\end{aligned} \tag{3.6}$$

By (3.4) once again and the fact that $C^{1,\kappa} \cap L^\infty$ is a multiplication algebra, it holds that

$$\begin{aligned}&\|\nabla III\|_{C^\kappa} \\ &\lesssim \int_{\frac{t+t'}{2}}^{t} (t-s)^{-\frac{1}{2\beta}} \Big(\|\nabla v(s)\|_{C^\kappa} \|P(s,t')f\|_{L^\infty} + \|v(s)\|_{L^\infty} \|\nabla P(s,t')f\|_{C^\kappa} \Big) ds\end{aligned}$$

$$\lesssim \int_{\frac{t+t'}{2}}^{t} (t-s)^{-\frac{1}{2\beta}} \Big(s^{-\frac{\beta+\alpha+\kappa+2}{2\beta}+\frac{\epsilon}{2}} (t')^{-1+\tau-\frac{\epsilon}{2}} \|f\|_{Y^\alpha} + \|v(s)\|_{L^\infty} \|\nabla P(s,t')f\|_{C^\kappa} \Big) ds \tag{3.7}$$

where the first term in the integrand would be $O((t')^{-1+\tau-\frac{\epsilon}{2}} t^{-\frac{\alpha+\kappa+3-\beta}{2\beta}+\frac{\epsilon}{2}} \|f\|_{Y^\alpha})$. Hence, defining $h(t) := (t-t')^{\frac{1}{2\beta}} t^{\frac{1+\kappa}{2\beta}} \|\nabla P(t,t')\|_{C^\kappa}$ and summing (3.5)–(3.7), we then have

$$h(t) \lesssim (t')^{-1+\tau} (\frac{t}{t'})^{\frac{\epsilon}{2}} \|f\|_{Y^\alpha} + \int_{\frac{t+t'}{2}}^{t} (t-s)^{-\frac{1}{2\beta}} \|v(s)\|_{L^\infty} h(s) ds.$$

Finally, we conclude once again by Lemma C.2, (2.26) and (2.27) to gain the desired result. □

3.2. Fixed point argument

We now define the subcritical sapce for the perturbation $\{\omega^{(i)}\}_{i\in\{1,2\}}$:

$$X^\alpha = \{\omega \in (C^0((0,1]; C^{1,\kappa}(\mathbb{T}^2;\mathbb{R}^2))\},$$

with norm

$$\|\omega\|_{X^\alpha} = \sup_{t\in(0,1]} (t^{\frac{1}{2}-\frac{\tau}{2}} \|\omega\|_{L^\infty} + t^{\frac{1}{2}+\frac{1}{2\beta}-\frac{\tau}{2}} \|\nabla\omega\|_{C^\kappa}) < \infty.$$

Since by Proposition 2.8, we have showed that the subcritical residual terms $\{F^{(i)}\}_{i\in\{1,2\}}$ can be arbitrary small. Therefore together with the fixed point theorem and this fact with the zero initial data, we can construct the perturbation $\{\omega^{(i)}\}_{i\in\{1,2\}}$.

Proposition 3.2. *Let A be sufficiently large. There exists $C' > 0$ such that for all $\eta_1 \in (0, C'^{-1})$ and $i \in \{1,2\}$, there exist $\omega^{(i)} \in B_X(0,\eta_1)$ such that*

$$\tilde{v}^{(i)} = v^{(i)} + \omega^{(i)},$$

is solution of (1.1). Furthermore, $\omega^{(i)} \to 0$ in $C^{(-1+\rho)}$ as $t \to 0$ with $\rho < \beta\tau - \alpha$.

Proof. Without loss of generality, we write v, ω and F to represent $v^{(i)}$,$\omega^{(i)}$ and $F^{(i)}$ with $i \in \{1,2\}$. A necessary condition for $\tilde{v} = v + \omega$ to solve the (1.1) with initial data $v(0)$ is that the perturbation ω must satisfy

$$\begin{cases} (\partial_t + (-\Delta)^\beta)\omega + \mathbb{P}\mathrm{div}\,(v\otimes\omega + \omega\otimes v) = \mathbb{P}\mathrm{div}\,(F - \omega\otimes\omega), \\ \omega(0,\cdot) = 0. \end{cases}$$

with F is defined as in Proposition 2.8. Recall that $P(t,t')$ is the linearized semigroup around v, defined in Subsection 3.1. We thus solve for ω as a fixed point of the $Y^\alpha \to X^\alpha$ map $\omega \mapsto T^1(\omega)$, where the map defined by:

$$\omega \mapsto T^1(\omega)(t) = \int_0^t P(t,t')(F - \omega\otimes\omega)(t') dt'.$$

Firstly, we prove that T_1 is a well-defined operator on X^α. Combining with the definition of X^α and Y^α, we have the elementary product estimate

$$\|\omega\otimes\omega\|_{Y^\alpha} \lesssim \sup_{0<t\le 1} (t^{1+\frac{1}{2\beta}-\tau} \|\omega(t)\|_{L^\infty} \|\nabla\omega(t)\|_{C^\kappa} + t^{1-\tau} \|\omega(t)\|_{L^\infty}^2) \lesssim \|\omega\|_{X^\alpha}^2.$$

Next we have

$$\begin{aligned}\|T^1(\omega)(t)\|_{L^\infty} &\lesssim \int_0^t (t')^{-1+\tau-\epsilon} t^{-\frac{1}{2\beta}+\epsilon} dt' \|F-\omega\otimes\omega\|_{Y^\alpha} \\ &\lesssim t^{-\frac{1}{2\beta}+\epsilon} \int_0^t (t')^{-1+\tau-\epsilon} dt' (\|\omega\|_{X^\alpha}^2 + \|F\|_{Y^\alpha}) \\ &\lesssim t^{-\frac{1}{2\beta}+\tau} (\|\omega\|_X^2 + \eta_0).\end{aligned}$$

and

$$\begin{aligned}\|\nabla T^1(\omega)(t)\|_{C^\kappa} &\lesssim t^{-\frac{1+\kappa}{2\beta}+\epsilon} \int_0^t (t')^{-1+\tau-\epsilon}(t-t')^{-\frac{1}{2\beta}} dt' (\|\omega\|_{X^\alpha}^2 + \|F\|_{Y^\alpha}) \\ &\lesssim t^{-\frac{2+\kappa}{2\beta}+\tau} (\|\omega\|_{X^\alpha}^2 + \eta_0).\end{aligned}$$

where we use Propostion 2.8, 3.1 and the fact that $\epsilon < \tau$.

Choosing ϵ_0 in Proposition (2.8) sufficiently small and recall that $\kappa \le (\tau+1)\beta - 1$, we then obtain when $t \le 1$,

$$\|T^1(\omega)(t)\|_X \le O(\|\omega\|_X) + \frac{1}{2}\eta_1,$$

By a similar calculation,

$$\|T^1(\omega_1) - T^1(\omega_2)\|_{X^\alpha} \le \|\omega_1-\omega_2\|_{X^\alpha}(\|\omega_1\|_{X^\alpha} + \|\omega_2\|_{X^\alpha}).$$

Choosing C' sufficiently large to compensate for the implicit constants, it follows that T_1 is a contraction on the $B_X(0,\eta_1)$ for all $\eta_1 \in (0, C'^{-1})$, and we conclude by the banach fix point theorem.

Finally, we prove control of ω near the initial time using the Duhamel formula

$$\omega(t) = \int_0^t e^{(t'-t)(-\Delta)^\beta} \mathbb{P}\mathrm{div}\,(F - \omega\otimes\omega - v\otimes\omega - \omega\otimes v)(t') dt', \tag{3.8}$$

By (C.1), (2.26), Proposition 2.8, the interpolation inequality and the fact that $\|w\|_{X^\alpha} \le \eta_1$, we have

$$\begin{aligned}\|w(t)\|_{C^{-1+\rho}} &\lesssim \int_0^t \|(F - \omega\otimes\omega - v\otimes\omega - \omega\otimes v)(t')\|_{C^\rho} dt' \\ &\lesssim \int_0^t ((t')^{-1+\tau-\frac{\rho}{2\beta}} + (t')^{-1+\frac{\tau}{2}-\frac{\rho+\alpha}{2\beta}}) dt' \\ &\lesssim (t)^{\frac{\tau}{2}-\frac{\rho+\alpha}{2\beta}} \xrightarrow[t\to 0]{} 0,\end{aligned}$$

for all $t \in (0,1]$ and $\rho < \beta\tau - \alpha$.

□

4 Proof of the main results

Proof of Proposition 1.3. Recall from definitions, we can write

$$u^{(1)} \triangleq v^{(1)} + \omega^{(1)} = \sum_{k\ge 0\, even} v_k + \sum_{k \ge 0\, odd} \bar{v}_k + \omega^{(1)},$$

and

$$u^{(2)} \triangleq v^{(2)} + \omega^{(2)} = \sum_{k\geq 0\, odd} v_k + \sum_{k\geq 0\, even} \bar{v}_k + \omega^{(2)}.$$

In Section 3.2, we have shown that for $i \in \{1,2\}$, there exist $\omega^{(i)} \in C^0((0,1]; C^{1,\kappa})$ obeying suitable equations such that $\{u^{(i)}\}_{i\in\{1,2\}}$ obey (1.1). Moreover, by standard regularity theory, $\{u^{(i)}\}_{i\in\{1,2\}}$ are smooth on $(0,1] \times \mathbb{T}^2$.

Next we argue that both solutions attain the initial data V^0. As shown in Proposition 3.2, $\{\omega^{(i)}\}_{i\in\{0,1\}} \to 0$ in $C^{-1+\rho}(\mathbb{T}^2)$. Thus, $\{\omega^{(i)}\}_{i\in\{0,1\}}$ belongs to $C([0,1]; B^{-\beta-\alpha-\varepsilon'}_{\infty,\infty})$. It remains to show $\{v^{(i)}\}_{i\in\{0,1\}} \in C([0,1]; B^{-\beta-\alpha-\varepsilon'}_{\infty,\infty})$. Since $\{v^{(i)}\}_{i\in\{0,1\}} \in C^\infty((0,1] \times \mathbb{T}^2)$, it suffice to prove that $\{v^{(i)}\}_{i\in\{0,1\}}$ are continuous at $t = 0$ in $B^{-\beta-\alpha-\varepsilon'}_{\infty,\infty}$. Recalling the defintion of $v^{(i)}$ in Definition 2.6, we have

$$\|v^{(i)}(t) - V^0\|_{B^{-\beta-\alpha-\varepsilon'}_{\infty,\infty}} \lesssim \sum_{k\in\mathbb{N}} \|v^0_k\|_{B^{-\beta-\alpha-\varepsilon'}_{\infty,\infty}} \left((e^{-(2\pi N_k)^{2\beta} t} - 1) + (e^{-2\cdot(2\pi N_{k+1})^{2\beta} t} - 1) \right). \tag{4.1}$$

Following the proof of Proposition 2.5, one can infers that

$$\|v^0_k\|_{B^{-\beta-\alpha-\varepsilon'}_{\infty,\infty}} \lesssim N_k^{-\varepsilon'}, \qquad \forall k \in \mathbb{N}. \tag{4.2}$$

Substituting (4.2) into (4.1), we arrive at

$$\|v^{(i)}(t) - V^0\|_{B^{-\beta-\alpha-\varepsilon'}_{\infty,\infty}} \lesssim \sum_{k\leq M} N_k^{-\varepsilon'} \left((e^{-(2\pi N_k)^{2\beta} t} - 1) + (e^{-2\cdot(2\pi N_{k+1})^{2\beta} t} - 1) \right) + \sum_{k\geq M} N_k^{-\varepsilon'} \xrightarrow[t\to 0]{} 0,$$

where we first select large M to make the second term small, then fix M and takie t small to make the first term small as well.

Finally, we prove that the two solutions are distinct by considering a time scale when all but the lowest frequency mode should have dissipated away, for exmaple $t_0 = N_0^{-2\beta}$. It follows from the Proposition 3.2 that $\omega^{(i)} \in B_X(0, \eta_1)$. By the triangle inquality, we get

$$\begin{aligned} \|u^{(1)}(t_0) - u^{(2)}(t_0)\|_{L^\infty} \geq \|v_0\|_{L^\infty} &- \sum_{k\geq 1} \|v_k(t_0)\|_{L^\infty} - \sum_{k\geq 0} \|\bar{v}_k(t_0)\|_{L^\infty} \\ &- \|\omega^{(1)}(t_0)\|_{L^\infty} - \|\omega^{(2)}(t_0)\|_{L^\infty}. \end{aligned} \tag{4.3}$$

By (2.28), one obtains

$$\sum_{k\geq 1} \|v_k(t_0)\|_{L^\infty} \lesssim \sum_{k\geq 1} N_k^{\beta+\alpha} \exp\left(-N_k^{2\beta} N_0^{-2\beta}\right) \lesssim N_1^{\beta+\alpha} \exp\left(-N_1^{2\beta} N_0^{-2\beta}\right) \leq N_0^{\beta+\alpha} \left(\frac{N_1}{N_0}\right)^{-100}, \tag{4.4}$$

and

$$\sum_{k\geq 0} \|\bar{v}_k(t_0)\|_{L^\infty} \lesssim \sum_{k\geq 0} N_k^{\beta+\alpha} \exp\left(-N_{k+1}^{2\beta} N_0^{-2\beta}\right) \lesssim N_0^{\beta+\alpha} \exp\left(-N_1^{2\beta} N_0^{-2\beta}\right) \leq N_0^{\beta+\alpha} \left(\frac{N_1}{N_0}\right)^{-100}. \tag{4.5}$$

For the $\{\omega^{(i)}\}_{i\in\{1,2\}}$, we have by Proposition 3.2

$$\|\omega^{(i)}(t_0)\|_{L^\infty} \lesssim \eta_1 t_0^{-\frac{1}{2}+\frac{\tau}{2}} \lesssim N_0^{\beta+\alpha-\beta\tau}. \tag{4.6}$$

We now only need to establish the exact value of $v_0(t_0,x)$ on $L^\infty(\mathbb{T}^2)$, recalling from Definition 2.2, Definition 2.4 and Definition 2.6, we easily have

$$\|v_0(t_0,\cdot)\|_{L^\infty} = \varepsilon(2\pi)^2 N_0^{\beta+\alpha} e^{-(2\pi)^{2\beta}}, \tag{4.7}$$

Now together with (4.4)-(4.7), we then have

$$\|u^{(1)}(t_0) - u^{(2)}(t_0)\|_{L^\infty} \geq C^{-1} N_0^{\beta+\alpha} - A^{-c(b,\beta,\alpha,\tau)} N_0^{\beta+\alpha},$$

for a $c(b,\beta,\alpha,\tau) > 0$. By taking A sufficiently large, we conclude that $u^{(1)}$ and $u^{(2)}$ are distinct. □

Proof of Theorem 1.2. We will use the method of integral equations and the contraction mapping principle to prove the existence. To do this, we define

$$D = \{u : (0,1] \to L^\infty(\mathbb{T}^d); \nabla\cdot u = 0 \quad \text{and} \quad \sup_{t\in(0,1]} t^{\frac{\beta+\alpha}{2\beta}} \|u(t)\|_{L^\infty(\mathbb{T}^d)}\},$$

with norm

$$\|u\|_D = \sup_{t\in(0,1]} t^{\frac{\beta+\alpha}{2\beta}} \|u(t)\|_{L^\infty(\mathbb{T}^d)}.$$

The mild solution to 1.1 is defined as the fixed point of the operator

$$(Au)(t,x) = e^{-t(-\Delta)^\beta} u_0(x) - \int_0^t e^{-(t-s)(-\Delta)^\beta} \mathbb{P}\,\mathrm{div}(u\otimes u)(s,x)\,ds$$

where

$$e^{-t(-\Delta)^\beta} f(x) := K_t^\beta(x) * f(x) \quad \text{with} \quad \widehat{K_t^\beta}(\xi) = e^{-t|\xi|^{2\beta}}.$$

Now we aim to find a fixed point Au in D. To do this, we first estimate $\|Au(t)\|_{L^\infty(\mathbb{T}^d)}$ for $t\in(0,1]$.

$$\|Au(t)\|_{L^\infty(\mathbb{T}^d)} \leq \|e^{-t(-\Delta)^\beta} u_0\|_{L^\infty(\mathbb{T}^d)} + \int_0^t (t-s)^{-\frac{1}{2\beta}} \|u(s)\|^2_{L^\infty(\mathbb{T}^d)}\,ds.$$

Therefore, we have

$$\begin{aligned}
\|Au(t)\|_D &\leq \sup_{t\in(0,1]} t^{\frac{\beta+\alpha}{2\beta}} \|e^{-t(-\Delta)^\beta} u_0\|_{L^\infty(\mathbb{T}^d)} + \sup_{t\in(0,1]} t^{\frac{\beta+\alpha}{2\beta}} \int_0^t (t-s)^{-\frac{1}{2\beta}} \|u(s)\|_{L^\infty(\mathbb{T}^d)} \|u(s)\|_{L^\infty(\mathbb{T}^d)}\,ds \\
&\leq C_1 \|u_0\|_{\dot{B}^{-(\beta+\alpha)}_{\infty,\infty}(\mathbb{T}^d)} + \|u\|_D^2 \sup_{t\in(0,1]} t^{\frac{\beta+\alpha}{2\beta}} \int_0^t (t-s)^{-\frac{1}{2\beta}} s^{-\frac{\beta+\alpha}{\beta}}\,ds \\
&\leq C_1 \|u_0\|_{\dot{B}^{-(\beta+\alpha)}_{\infty,\infty}(\mathbb{T}^d)} + \|u\|_D^2 \sup_{t\in(0,1]} t^{\frac{\beta-\alpha-1}{2\beta}} \int_0^1 (1-s)^{-\frac{1}{2\beta}} s^{-\frac{\beta+\alpha}{\beta}}\,ds \\
&\leq C_1 \|u_0\|_{\dot{B}^{-\beta-\alpha}_{\infty,\infty}(\mathbb{T}^d)} + C_2 \|u\|_D^2.
\end{aligned}$$

where we use the fact that $\alpha<0$, $\alpha\leq\beta-1$ and $\beta>\frac{1}{2}$. We then get

$$\|Au - A\bar{u}\|_D \leq C_3 \max\{\|u\|_D, \|\bar{u}\|_D\} \|u-\bar{u}\|_D,$$

for any $u,\bar{u}\in D$. Then, we will prove that A is a contraction mapping on $B_D(0,r)$ with a suitable r. In fact, if there exists $\varepsilon_\beta>0$ small enough such that

$$\|e^{-t(-\Delta)^\beta} u_0\|_D \leq C_1 \|u_0\|_{\dot{B}^{-\beta-\alpha}_{\infty,\infty}(\mathbb{T}^n)} < \varepsilon_\beta \quad \text{and} \quad 6C_3C_1 \|u_0\|_{\dot{B}^{-\beta-\alpha}_{\infty,\infty}(\mathbb{T}^n)} < 1/2,$$

then, we can prove that for any $u, v \in B_Y(0,r)$ with $r = 2C_1\|u_0\|_{\dot{B}^{-(\beta+\alpha)}_{\infty,\infty}(\mathbb{T}^d)}$,

$$\begin{aligned}\|Au - Av\|_D &\leq C_3 \max\{\|u\|_D, \|v\|_D\}\|u-v\|_D \\ &\leq C_3 r\|u-v\|_D \\ &\leq \frac{1}{2}\|u-v\|_D.\end{aligned}$$

Thus, A is a contraction mapping in $B_D(0,r)$ and has a unique fixed point u.

□

A Tools from convex integration

In this section, we recall some useful tools which is developed in the convex integration method.

Lemma A.1. *There exists a set $\Lambda \subset \mathbb{S}^1 \cap \mathbb{Q}^2$ that satisfies $\Lambda = -\Lambda$ and consists of vectors ξ with associated orthonormal bases $(\xi, \bar{\xi})$ and smooth positive functions $\Gamma_\xi : B_{\frac{1}{7}}(\mathrm{Id}) \to \mathbb{R}$ satisfying $a_\xi = a_{-\xi}$, where $B_{\frac{1}{7}}(\mathrm{Id})$ is the ball of radius $\frac{1}{7}$ centered at Id in the space of 2×2 symmetric matrices, such that we have the following identity:*

$$S = \sum_{\xi\in\Lambda} \Gamma_\xi^2(S)\bar{\xi}\otimes\bar{\xi}, \quad \forall S \in B_{\frac{1}{7}}(\mathrm{Id}). \tag{A.1}$$

We recall the antidivergence operators $\mathcal{R}$ that are introduced in [13].

Proposition A.2. *There exists a linear operator $\mathcal{R} : C^\infty(\mathbb{T}^2;\mathbb{R}^2) \to C^\infty(\mathbb{T}^2; S^{2\times 2})$ satisfying*

$$\operatorname{div} \mathcal{R}f = f - \int_{\mathbb{T}^2} f, \quad \forall f \in C^\infty(\mathbb{T}^2;\mathbb{R}^2),$$

Lemma A.3. *Let $\mu > 0$, for any smooth function f and $m > 0$, we have*

$$\|\mathcal{R}(f(x)e^{2\pi i\lambda kx})\|_{C^\mu} \lesssim_{\mu,m} \lambda^{-1+\mu}\|f\|_{L^\infty} + \lambda^{-m+\mu}\|\nabla^m f\|_{L^\infty} + \lambda^{-m}\|\nabla^m f\|_{C^\mu}, \tag{A.2}$$

$$\tag{A.3}$$

Lemma A.4. *Let $\mathbb{S}^1$ be the unit circle and $\mathbb{Q}$ the set of rational numbers. Given $\Lambda \subseteq \mathbb{S}^1 \cap \mathbb{Q}^2$ such that $-\Lambda = \Lambda$. Then for any $b_k \in \mathbb{R}$ with $b_k = b_{-k}$, the vector field*

$$W(x) = \sum_{\xi\in\Lambda} b_\xi \mathrm{i}\bar{\xi}e^{2\pi\mathrm{i}\xi\cdot x},\ \xi \perp \bar{\xi},$$

is real-valued, divergence-free and satisfies

$$\operatorname{div}_x(W\otimes W) = \frac{1}{2}\nabla_x\left(|W|^2 - \left|\sum_{\xi\in\Lambda} b_\xi e^{2\pi\mathrm{i}\xi\cdot x}\right|^2\right).$$

B Fractional Leibniz estimate

Lemma B.1. *Let $P, Q, s > 0$ and $\xi \in \mathbb{T}^2\setminus\{0\}$. For any smooth functions f satisfying*

$$\|\nabla^n f\|_{L^\infty(\mathbb{T}^2)} \lesssim P^n, \qquad \forall n \in \mathbb{N}, \tag{B.1}$$

then we have

$$\|\nabla^m[(-\Delta)^s, f]\sin(2\pi Q\xi\cdot)\|_{L^\infty(\mathbb{T}^2)} \lesssim (Q+P)^m(Q^{2s-1}P^4 + P^{2s+3}), \tag{B.2}$$

for any $m \in \mathbb{N}$.

Proof. Taking the fourier series of f,

$$f(x) = \sum_{l\in\mathbb{Z}^2} \hat{f}(l)e^{2\pi i l\cdot x}.$$

By (B.1) and Parseval's identity, we obtain

$$\|\nabla^m f\|_{L^2(\mathbb{T}^2)} = \left(\sum_{l\in\mathbb{Z}^2} |l|^{2m}|\hat{f}(l)|^2\right)^{\frac{1}{2}} \lesssim \|\nabla^m f\|_{L^\infty(\mathbb{T}^2)} \lesssim P^m, \qquad \forall m \in \mathbb{N},$$

which immediately implies

$$|l|^m|\hat{f}(l)| \lesssim P^m, \qquad \forall (l,m) \in \mathbb{Z}^2\times\mathbb{N}. \tag{B.3}$$

we compute

$$\begin{aligned}
[(-\Delta)^s, f(x)]\sin(2\pi Q\xi\cdot x) &= (-\Delta)^s(f(x)\sin(2\pi Q\xi\cdot x)) - f(x)(-\Delta)^s(\sin(2\pi Q\xi\cdot x)) \\
&= \sum_{l\in\mathbb{Z}^2}(2\pi)^{2s}\left(|l+Q\xi|^{2s} - |Q\xi|^{2s}\right)\hat{f}(l)e^{2\pi i(l+Q\xi)\cdot x} \\
&= \left(\sum_{0<|l|\le|Q\xi|} + \sum_{|l|>|Q\xi|}\right)(2\pi)^{2s}\left(|l+Q\xi|^{2s} - |Q\xi|^{2s}\right)\hat{f}(l)e^{2\pi i(l+Q\xi)\cdot x} \\
&\triangleq L_1 + L_2.
\end{aligned}$$

For L_1, we estimate

$$\begin{aligned}
\|L_1\|_{L^\infty(\mathbb{T}^2)} &\lesssim \sum_{0<|l|\le|Q\xi|} Q^{2s}\left(|\frac{l}{Q}+\xi|^{2s} - |\xi|^{2s}\right)|\hat{f}(l)| \\
&\lesssim \sum_{0<|l|\le|Q\xi|} Q^{2s}\cdot|\frac{l}{Q}||\hat{f}(l)| \\
&\lesssim Q^{2s-1}\max_l |l|^4|\hat{f}(l)|\sum_{l\neq 0}\frac{1}{|l|^3} \\
&\lesssim Q^{2s-1}P^4.
\end{aligned} \tag{B.4}$$

For L_2, we get

$$\|L_2\|_{L^\infty(\mathbb{T}^2)} \lesssim \sum_{|l|>|Q\xi|}|l|^{2s}|\hat{f}(l)| \lesssim \max_l |l|^{2s+3}|\hat{f}(l)|\sum_{l\neq 0}\frac{1}{|l|^3} \lesssim P^{2s+3}. \tag{B.5}$$

Combining (B.4) and (B.5) together, we obtain (B.2) for the case $m=0$. For $m \ge 1$, consider the case $m = 1$ as an example. We have

$$\nabla[(-\Delta)^s, f]\sin(2\pi Q\xi\cdot) = [(-\Delta)^s, \nabla f]\sin(2\pi Q\xi\cdot) + [(-\Delta)^s, f]\nabla(\sin(2\pi Q\xi\cdot)).$$

By a similar argument as above, the desired estimates follow. So we omit the details. $\square$

C Grönwall inequality and heat esitmate

We recall the fractional Grönwall inequality from [10, Lemma B.2].

Lemma C.1. *Let $f(t)$, $h_i(t)$, $b_i(t)$, and $a(t)$ be non-negative continuous functions on $[t_0,\infty)$, with $b_i(t)$ and $a(t)$ non-decreasing. Suppose for some $p>1$,*

$$f(t) \le a(t) + b_1(t)\int_{t_0}^t h_1(s)f(s)\,ds + \left(b_2(t)\int_{t_0}^t h_2(s)f(s)^p\,ds\right)^{\frac{1}{p}}.$$

Then for any constant C_6 large enough so that $C_6^{-1}+(C_6p)^{-\frac{1}{p}}<1$, we have

$$f(t) \le C_6 a(t)\exp\left(C_6\left(b_1(t)\int_{t_0}^t h_1(s)\,ds + b_2(t)\int_{t_0}^t h_2(s)\,ds\right)\right).$$

Lemma C.2. *If $a(t)$, $f(t)$, $g_1(t)$, and $g_2(t)$ are positive continuous functions on $[t_0,\infty)$ with $a(t)$ non-decreasing and*

$$f(t) \le a(t) + \int_{t_0}^t \big(g_1(s) + (t-s)^{-\frac{1}{2\beta}}g_2(s)\big)f(s)\,ds \quad \forall t\ge t_0,$$

then for any $p>2$,

$$f(t) \lesssim_p a(t)\exp\Big(O_p\Big(\int_{t_0}^t g_1(s)\,ds + \|s^{\frac{1}{2\beta}}g_2\|_{L^\infty([t_0,t])}^{p-\frac{2\beta}{2\beta-1}}\int_{t_0}^t g_2^{\frac{2\beta}{2\beta-1}}(s)\,ds\Big)\Big) \quad \forall t\ge t_0.$$

Proof. By Hölder inequality,

$$\int_{t_0}^t (t-s)^{-\frac{1}{2\beta}}g_2(s)f(s)ds \le \left(\int_{t_0}^t (t-s)^{-\frac{p'}{2\beta}}g_2^{(1-\frac{2\beta}{(2\beta-1)p})p'}(s)ds\right)^{\frac{1}{p'}}\left(\int_{t_0}^t g_2^{\frac{2\beta}{2\beta-1}}(s)f^p(s)\right)^{\frac{1}{p}}$$

Since we have

$$\begin{aligned}\left(\int_{t_0}^t (t-s)^{-\frac{p'}{2\beta}}g_2^{(1-\frac{2\beta}{(2\beta-1)p})p'}(s)ds\right)^{\frac{1}{p'}} &\le \|s^{\frac{2\beta-1}{2\beta}}g_2(s)\|_{L^\infty}^{(1-\frac{2\beta}{(2\beta-1)p})}\left(\int_{t_0}^t (t-s)^{-\frac{p'}{2\beta}}s^{(-\frac{2\beta-1}{2\beta}+\frac{1}{p})p'}ds\right)^{\frac{1}{p'}}\\ &\le O_p\left(\|s^{\frac{2\beta-1}{2\beta}}g_2(s)\|_{L^\infty}^{(1-\frac{2\beta}{(2\beta-1)p})}\right)\end{aligned}$$

Then, we get that

$$f(t) \le a(t) + \int_{t_0}^t g_1(s)f(s)ds + O_p\left(\|s^{\frac{2\beta-1}{2\beta}}g_2(s)\|_{L^\infty}^{(1-\frac{2\beta}{(2\beta-1)p})}\left(\int_{t_0}^t g_2^{\frac{2\beta}{2\beta-1}}(s)f^p(s)\right)^{\frac{1}{p}}\right)$$

Together with Lemma C.1, this finishes the proof. □

Now recall the heat estimate.

Lemma C.3. *For all $m\ge 0$, $s_1,s_2\in\mathbb{R}$ satisfying $m+s_1-s_2\ge 0$, and $f\in\mathcal{C}^{s_2}$, we have*

$$\|e^{-t(-\Delta)^\beta}\nabla^m f\|_{\mathcal{C}^{s_1}} \lesssim_{m,s_1,s_2} t^{-\frac{m+s_1-s_2}{2\beta}}\|f\|_{\mathcal{C}^{s_2}} \tag{C.1}$$

where $\mathcal{C}^s$ is the homogeneous Höder-Zygmund space with norm

$$\|u\|_{\mathcal{C}^s} := \sup_{N\in 2^{\mathbb{N}}} N^s\|P_N u\|_{L^\infty} < \infty.$$

Moreover, the above estimate holds as well if f on the left-hand side is replaced by $\mathbb{P}f$. We remark that for $s = m + \alpha$ with $m \in \mathbb{N}$ and $\alpha \in (0,1)$, $\mathcal{C}^s$ coincides with the standard homogeneous Höder space C^s.

Acknowledgments This work was partially supported by the National Natural Science Foundation of China (No.12571261).

References

[1] D. Albritton, E. Brué, and M. Colombo. Non-uniqueness of Leray solutions of the forced Navier-Stokes equations. *Ann. of Math. (2)*, 196(1):415–455, 2022.

[2] D. Albritton and M. Colombo. Non-uniqueness of Leray solutions to the hypodissipative Navier-Stokes equations in two dimensions. *Comm. Math. Phys.*, 402(1):429–446, 2023.

[3] J. Bourgain and N. s. Pavlović. Ill-posedness of the Navier-Stokes equations in a critical space in 3D. *J. Funct. Anal.*, 255(9):2233–2247, 2008.

[4] T. Buckmaster, M. Colombo, and V. Vicol. Wild solutions of the Navier-Stokes equations whose singular sets in time have Hausdorff dimension strictly less than 1. *J. Eur. Math. Soc. (JEMS)*, 24(9):3333–3378, 2022.

[5] T. Buckmaster, C. de Lellis, L. Székelyhidi, Jr., and V. Vicol. Onsager's conjecture for admissible weak solutions. *Comm. Pure Appl. Math.*, 72(2):229–274, 2019.

[6] T. Buckmaster and V. Vicol. Nonuniqueness of weak solutions to the Navier-Stokes equation. *Ann. of Math. (2)*, 189(1):101–144, 2019.

[7] M. Cannone. *Ondelettes, paraproduits et Navier-Stokes*. Diderot Editeur, Paris, 1995. With a preface by Yves Meyer.

[8] A. Cheskidov and X. Luo. Sharp nonuniqueness for the Navier-Stokes equations. *Invent. Math.*, 229(3):987–1054, 2022.

[9] A. Cheskidov and X. Luo. L^2-critical nonuniqueness for the 2D Navier-Stokes equations. *Ann. PDE*, 9(2):Paper No. 13, 56, 2023.

[10] M. P. Coiculescu and S. Palasek. Non-uniqueness of smooth solutions of the navier–stokes equations from critical data. *Inventiones Mathematicae*, 2025.

[11] C. De Lellis and L. Székelyhidi, Jr. The Euler equations as a differential inclusion. *Ann. of Math. (2)*, 170(3):1417–1436, 2009.

[12] C. De Lellis and L. Székelyhidi, Jr. On admissibility criteria for weak solutions of the Euler equations. *Arch. Ration. Mech. Anal.*, 195(1):225–260, 2010.

[13] C. De Lellis and L. Székelyhidi, Jr. Dissipative continuous Euler flows. *Invent. Math.*, 193(2):377–407, 2013.

[14] H. Fujita and T. Kato. On the Navier-Stokes initial value problem. I. *Arch. Rational Mech. Anal.*, 16:269–315, 1964.

[15] P. Germain. The second iterate for the Navier-Stokes equation. *J. Funct. Anal.*, 255(9):2248–2264, 2008.

[16] V. Giri and R. a.-O. Radu. The Onsager conjecture in 2D: a Newton-Nash iteration. *Invent. Math.*, 238(2):691–768, 2024.

[17] E. Hopf. über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen. *Math. Nachr.*, 4:213–231, 1951.

[18] P. Isett. A proof of Onsager's conjecture. *Ann. of Math. (2)*, 188(3):871–963, 2018.

[19] H. Jia and V. Sverak. Are the incompressible 3d Navier-Stokes equations locally ill-posed in the natural energy space? *J. Funct. Anal.*, 268(12):3734–3766, 2015.

[20] H. Jia and V. ˇ Sverák. Local-in-space estimates near initial time for weak solutions of the Navier-Stokes equations and forward self-similar solutions. *Invent. Math.*, 196(1):233–265, 2014.

[21] T. Kato. Strong L^p-solutions of the Navier-Stokes equation in $\mathbf{R}^m$, with applications to weak solutions. *Math. Z.*, 187(4):471–480, 1984.

[22] H. Koch and D. Tataru. Well-posedness for the Navier-Stokes equations. *Adv. Math.*, 157(1):22–35, 2001.

[23] H. Kozono and H. Sohr. Remark on uniqueness of weak solutions to the Navier-Stokes equations. *Analysis*, 16(3):255–271, 1996.

[24] O. A. Ladyˇ zenskaja. Uniqueness and smoothness of generalized solutions of Navier-Stokes equations. *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 5:169–185, 1967.

[25] J. Leray. Sur le mouvement d'un liquide visqueux emplissant l'espace. *Acta Math.*, 63(1):193–248, 1934.

[26] Y. Li, P. Qu, Z. Zeng, and D. Zhang. Sharp non-uniqueness for the 3D hyperdissipative Navier-Stokes equations: beyond the Lions exponent. *J. Math. Pures Appl. (9)*, 190:Paper No. 103602, 64, 2024.

[27] T. Luo and E. S. Titi. Non-uniqueness of weak solutions to hyperviscous Navier-Stokes equations: on sharpness of J.-L. Lions exponent. *Calc. Var. Partial Differential Equations*, 59(3):Paper No. 92, 15, 2020.

[28] C. Miao, Y. Nie, and W. Ye. Non-uniqueness of smooth solutions to the navier-stokes equations on torus $\mathbb{T}^2$, 2026.

[29] S. Palasek. Non-uniqueness in the leray–hopf class for a dyadic navier–stokes model. *International Mathematics Research Notices*, 2025(22):rnaf344, 2025.

[30] F. Planchon. Asymptotic behavior of global solutions to the Navier-Stokes equations in $\mathbf{R}^3$. *Rev. Mat. Iberoamericana*, 14(1):71–93, 1998.

[31] G. Prodi. Un teorema di unicità per le equazioni di Navier-Stokes. *Ann. Mat. Pura Appl. (4)*, 48:173–182, 1959.

[32] J. Serrin. On the interior regularity of weak solutions of the Navier-Stokes equations. *Arch. Rational Mech. Anal.*, 9:187–195, 1962.

[33] M. Vishik. Instability and non-uniqueness in the cauchy problem for the euler equations of an ideal incompressible fluid. part i, 2018.

[34] M. Vishik. Instability and non-uniqueness in the cauchy problem for the euler equations of an ideal incompressible fluid. part ii, 2018.

[35] B. Wang. Ill-posedness for the Navier-Stokes equations in critical Besov spaces $\dot{B}^{-1}_{\infty,q}$. *Adv. Math.*, 268:350–372, 2015.

[36] T. Yoneda. Ill-posedness of the 3D-Navier-Stokes equations in a generalized Besov space near BMO^{-1}. *J. Funct. Anal.*, 258(10):3376–3387, 2010.