

An Exact Hypergeometric Tail Certificate for Ramanujan Challenge Problem 2.8

Problem 2.8 submission

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Abstract

Let $G_N = M_0 M_1 \cdots M_{N-1}$ be the 4×4 transfer product in Ramanujan Challenge Problem 2.8, and let $P_{N,j}$ and $Q_{N,j}$ be the two official seeded rows evaluated in column j . We prove

$$\lim_{N \rightarrow \infty} \frac{P_{N,j}}{Q_{N,j}} = \frac{\sqrt{10005}}{\pi} \quad (j = 1, 2, 3, 4).$$

Equivalently, $Q_{N,j}/P_{N,j} \rightarrow \pi/\sqrt{10005}$.

The missing connection constant is fixed by an exact rank-three hypergeometric tail. A nonterminating ${}_4F_3$ Euler jet is carried backward by the authoritative matrix, while the first denominator is a terminating adjoint ${}_4F_3$. Their common differential gauge gives an all- N Padé divisibility theorem. Positivity of the terminating denominator at the negative CM point, together with a balanced-transfer Cauchy estimate, turns that formal divisibility into a direct fixed-point convergence proof. The symbolic contiguity and adjoint identities are included as reproducible Wolfram Language and SageMath certificates.

Contents

1	Statement and compact form of the seeds	2
2	The CM function and its exact value	3
3	The nonterminating adjoint tail	3
4	The rank-three error carrier	4
5	A discrete valuation lemma	4
6	The terminating denominator	6
7	From formal contact to convergence at x_0	7
8	Identification of the first-column limit	8
9	The other three official columns	8
10	Reproducibility map	10
	References	10

1 Statement and compact form of the seeds

Put

$$R = 151931373056001 = 53360^3 + 1, \quad x_0 = \frac{1}{R}, \quad z = -\frac{x}{1-x}.$$

Thus the official CM point is

$$z_0 = -\frac{1}{R-1} = -\frac{1}{53360^3}.$$

Let

$$G_N = M_0 M_1 \cdots M_{N-1}, \quad G_0 = I_4,$$

where $M_N = M(N, x)$ is the authoritative transfer matrix in the analytic deformation

$$236337691420383 \mapsto \frac{14/x - 567}{9}.$$

At $x = x_0$, this is the exact identity $236337691420383 = (14R - 567)/9$. Thus every later use of Cauchy's theorem concerns this explicitly defined rational x -family. The complete entries of $M(N, x)$ appear verbatim in the accompanying CAS certificates.

Define four Pascal rows

$$\begin{aligned} b_0 &= (1, 0, 0, 0), & b_1 &= (1, 1, 0, 0), \\ b_2 &= (1, 2, 1, 0), & b_3 &= (1, 3, 3, 1) \end{aligned}$$

and let $\mathcal{P}$ be the matrix with rows b_0, b_1, b_2, b_3 . The compact denominator row is

$$C(x) = \left(\frac{18}{x} + \frac{159}{4}, \frac{54}{x} + \frac{131}{2}, \frac{54}{x} + 27, \frac{18}{x} \right).$$

Equivalently,

$$C = \frac{18}{x}b_3 + \frac{5}{4}b_0 + \frac{23}{2}b_1 + 27b_2.$$

Set

$$A = 13591409, \quad B = 545140134, \quad S = 426880.$$

The two official initial rows have the exact form

$$A_1 = SC, \quad A_0 = AC - \frac{5}{4}H_0, \quad H_0 = Ab_0 + Bb_1 = (A + B, B, 0, 0). \quad (1)$$

At $x = x_0$, these identities reproduce the official integer rows entry by entry.

For $j = 1, \dots, 4$, write

$$P_{N,j} = A_0 G_N \mathbf{e}_j, \quad Q_{N,j} = A_1 G_N \mathbf{e}_j.$$

We first identify the first-column limit and then invoke the exact cyclic frame to cover all four columns.

2 The CM function and its exact value

Let

$$y(z) = {}_3F_2\left(\begin{matrix} \frac{1}{6}, \frac{1}{2}, \frac{5}{6} \\ 1, 1 \end{matrix}; z\right), \quad \theta = z \frac{d}{dz} = (1-x)x \frac{d}{dx},$$

and define

$$\Phi(x) = \frac{Ay(z) + B\theta y(z)}{S}. \quad (2)$$

The classical Chudnovsky identity is

$$\frac{1}{\pi} = \frac{12}{640320^{3/2}} \sum_{k=0}^{\infty} \frac{(6k)!}{(3k)!(k!)^3} (A + Bk) (-640320^{-3})^k.$$

The elementary coefficient identity

$$\frac{(6k)!}{(3k)!(k!)^3} = 1728^k \frac{(\frac{1}{6})_k (\frac{1}{2})_k (\frac{5}{6})_k}{(k!)^3}$$

and $640320 = 12 \cdot 53360$ give

$$Ay(z_0) + B\theta y(z_0) = \frac{640320^{3/2}}{12\pi} = \frac{426880\sqrt{10005}}{\pi}.$$

Consequently,

$$\boxed{\Phi(x_0) = \frac{\sqrt{10005}}{\pi}}. \quad (3)$$

3 The nonterminating adjoint tail

Put $n = N + 1$ and $\delta_N = \theta - n$. Define

$$F_N(z) = \kappa_N z^n {}_4F_3\left(\begin{matrix} n, n + \frac{1}{6}, n + \frac{1}{2}, n + \frac{5}{6} \\ 2n, 2n, 2n \end{matrix}; z\right), \quad (4)$$

where

$$\kappa_0 = \frac{5}{72}, \quad \frac{\kappa_{N+1}}{\kappa_N} = -\frac{(6N+7)(6N+11)}{576(N+1)^2(2N+3)^2}.$$

Its Euler jet is

$$k_N = \left(F_N, \delta_N F_N, \delta_N^2 F_N, \delta_N^3 F_N\right)^T.$$

Proposition 1 (Exact tail contiguity). *For every $N \geq 0$,*

$$\boxed{M_N k_{N+1} = k_N}. \quad (5)$$

Proof. The first row is verified coefficientwise from the ratio of consecutive ${}_4F_3$ coefficients. For the other rows, let $t = \delta_{N+1}$. The shifted tail satisfies

$$\left[(1-x)t(t+u)^3 + x(t+n+1)(t+n+\frac{7}{6})(t+n+\frac{3}{2})(t+n+\frac{11}{6})\right] F_{N+1} = 0,$$

where $u = 2N + 3$. Each of the remaining three row differences is divided by this degree-four Ore polynomial; its remainder is identically zero in $\mathbb{Q}(N, x)[t]$. The exact coefficient identity, the three Ore divisions, and the normalization ratio are checked in `p28_full_closure_certificate.wl` and `p28_kernel_contiguity_certificate.sage`. $\square$

For $N = 0$, the standard ascension identity gives

$$F_0 = y - 1 = \frac{5}{72} z {}_4F_3 \left(\begin{matrix} 1, \frac{7}{6}, \frac{3}{2}, \frac{11}{6} \\ 2, 2, 2 \end{matrix}; z \right). \quad (6)$$

Because $\mathcal{P}$ is the Pascal matrix,

$$\mathcal{P}k_0 = (y - 1, \theta y, \theta^2 y, \theta^3 y)^T.$$

4 The rank-three error carrier

Set

$$f = (y - 1, \theta y, \theta^2 y, \theta^3 y)^T, \quad \mathcal{E}_0 = \frac{5}{4} \mathcal{P} + fC, \quad \mathcal{E}_N = \mathcal{E}_0 G_N.$$

The transformed hypergeometric equation is

$$72\theta^3 y + 108x\theta^2 y + 46x\theta y + 5xy = 0. \quad (7)$$

Using the displayed decomposition of C , equations (6)–(7) give

$$Ck_0 = -\frac{5}{4}.$$

It follows that

$$\mathcal{E}_0 k_0 = \frac{5}{4} f + f(Ck_0) = 0.$$

Proposition 1 therefore implies

$$\boxed{\mathcal{E}_N k_N = 0 \quad (N \geq 0)}. \quad (8)$$

The same differential equation gives the exact row relation

$$72(\mathcal{E}_N)_{3,*} + 108x(\mathcal{E}_N)_{2,*} + 46x(\mathcal{E}_N)_{1,*} + 5x(\mathcal{E}_N)_{0,*} = 0. \quad (9)$$

5 A discrete valuation lemma

Let

$$H = \text{diag}(x, 1, 1, 1), \quad J_N = HM_N.$$

Every entry of J_N is regular at $x = 0$. If $u = 2N + 3$ and

$$a_N = \frac{144(u-1)^2}{u(3u-2)(3u+2)}, \quad V_N = (u^3, 3u^2, 3u, 1),$$

direct substitution in the authoritative matrix gives

$$J_N(0) = \begin{pmatrix} a_N \\ -1 \\ -1 \\ -1 \end{pmatrix} V_N. \quad (10)$$

Thus $J_N(0)$ has rank one.

The first nonconstant coefficient of the ${}_4F_3$ in (4) equals

$$c_N = \frac{u(3u-2)(3u+2)}{144(u-1)^2} = a_N^{-1}.$$

Since $z = -x + O(x^2)$,

$$Hk_N = x^{N+2}\eta_N \left[\begin{pmatrix} 1 \\ -c_N \\ -c_N \\ -c_N \end{pmatrix} + O(x) \right], \quad \eta_N \neq 0. \quad (11)$$

The leading vector in (11) is precisely the image direction in (10).

Lemma 1 (DVR step, including the extra first-column zero). *Let $R_0 = \mathbb{Q}[[x]]$, $H = \text{diag}(x, 1, 1, 1)$, and suppose*

$$E = x^N LH, \quad L \in \text{Mat}_4(R_0), \quad Ek = 0.$$

Assume $J = HM \in \text{Mat}_4(R_0)$, $Mk^+ = k$, and

$$\begin{aligned} k^+ &= x^r \alpha(\mathbf{e}_1 + xs + O(x^2)), \\ Hk &= x^r \beta(v_0 + xv_1 + O(x^2)), \end{aligned}$$

with $\alpha\beta \neq 0$. If $J(0)$ has rank one, $\text{im } J(0) = \mathbb{Q}v_0$, and $J(0)\mathbf{e}_1 \neq 0$, then

$$EM = x^{N+1}L^+H$$

for some $L^+ \in \text{Mat}_4(R_0)$.

Proof. Absorb β/α into v_0, v_1 . From $Jk^+ = Hk$,

$$J(\mathbf{e}_1 + xs + O(x^2)) = v_0 + xv_1 + O(x^2).$$

Write $L = L_0 + xL_1 + \dots$. The equation $L(Hk) = 0$ gives

$$L_0v_0 = 0, \quad L_0v_1 + L_1v_0 = 0.$$

Since $J(0)$ has image $\mathbb{Q}v_0$, $L_0J(0) = 0$, so LJ is entrywise divisible by x . The coefficient of x in its first column is

$$L_0(v_1 - J(0)s) + L_1v_0 = -L_0J(0)s = 0.$$

Hence that column is divisible by x^2 . Therefore

$$L^+ = x^{-1}LJH^{-1}$$

is regular and $EM = x^{N+1}L^+H$. □

Proposition 2 (All- N Padé divisibility). *For every $N \geq 0$, there is $L_N \in \text{Mat}_4(\mathbb{Q}[[x]])$ such that*

$$\boxed{\mathcal{E}_N = x^N L_N H.} \quad (12)$$

Thus every row of $\mathcal{E}_N$ has componentwise valuations at least

$$(N+1, N, N, N).$$

The last row has the stronger valuations

$$(N+2, N+1, N+1, N+1).$$

Proof. Every component of f is $O(x)$, while C has only a simple pole. All four components of f have leading term $-5x/72$. Consequently the constant term in the first column of fC is $-5/4$, cancelling the first component of every row of $(5/4)\mathcal{P}$. Hence $\mathcal{E}_0 = L_0H$.

Apply Lemma 1 inductively, using (8), (10), (11), and $M_N k_{N+1} = k_N$. The stronger last-row assertion follows from (9). $\square$

6 The terminating denominator

For the first column put

$$q_N(x) = CG_N \mathbf{e}_1, \quad n = N + 1, \quad Q_N(x) = x^n q_N(x),$$

and define

$$\widehat{Q}_N(z) = (1 - z)^n Q_N\left(-\frac{z}{1 - z}\right).$$

Since $x = -z/(1 - z)$, this is also the first component of $(-z)^n CG_N$.

Proposition 3 (Exact terminating denominator). *For every $N \geq 0$,*

$$\frac{\widehat{Q}_N(z)}{\alpha_n} = {}_4F_3\left(\begin{matrix} -n, -n - \frac{1}{6}, -n - \frac{1}{2}, -n - \frac{5}{6} \\ 1 - 2n, 1 - 2n, 1 - 2n \end{matrix}; z\right), \quad (13)$$

where

$$\alpha_1 = 18, \quad \frac{\alpha_{n+1}}{\alpha_n} = \frac{576n^2(2n+1)^2}{(6n+1)(6n+5)}. \quad (14)$$

Here and below the hypergeometric expression denotes the unambiguous finite sum over $0 \leq k \leq n$; it terminates before any lower Pochhammer symbol can vanish.

Proof. The proof is an exact differential-gauge calculation in $\mathbb{Q}(n, z)$. The nonterminating tail

$${}_4F_3\left(\begin{matrix} n, n + \frac{1}{6}, n + \frac{1}{2}, n + \frac{5}{6} \\ 2n, 2n, 2n \end{matrix}; z\right)$$

has a 4×4 Euler companion system. Direct simplification gives

$$\mathcal{C}_n(z) [-zM(2n+1, -z/(1-z))] - \theta [-zM(2n+1, -z/(1-z))] - [-zM(2n+1, -z/(1-z))] \mathcal{C}_{n+1}(z) = 0.$$

The transformed seed $-zC(-z/(1-z))$ is a horizontal adjoint row. Eliminating its other three coordinates from the horizontal equation produces exactly

$$\left[\theta(\theta - 2n)^3 - z(\theta - n)(\theta - n - \frac{1}{6})(\theta - n - \frac{1}{2})(\theta - n - \frac{5}{6})\right] \widehat{Q}_N = 0.$$

The analytic solution normalized at $z = 0$ is the terminating ${}_4F_3$ in (13).

For completeness, the CAS certificate does not rely only on this differential equation. It computes the actual one-step scalar operator and verifies its generic coefficient identity, its $k = 0$ normalization, and the separate top boundary $k = n + 1$. Every remainder simplifies identically to zero. This proves the statement for all n , not merely for sampled values. $\square$

Corollary 1 (Positivity at the CM point). *At $z_0 = -1/53360^3$,*

$$\widehat{Q}_N(z_0) \geq \alpha_n > 0.$$

Moreover,

$$\alpha_n \geq 18 \cdot 29^N (N!)^2.$$

Proof. For $0 \leq k \leq n$, the coefficient of z^k in (13) has sign $(-1)^k$. Since $z_0 < 0$, every summand is nonnegative. Also

$$\frac{576n^2(2n+1)^2}{(6n+1)(6n+5)} - 29n^2 = \frac{n^2(1260n^2 + 1260n + 431)}{(6n+1)(6n+5)} > 0.$$

Iterating (14) proves the lower bound. $\square$

7 From formal contact to convergence at x_0

This step is included to rule out a beyond-all-orders ambiguity. For $r = 0, 1$, let

$$E_{N,r}(x) = (\mathcal{E}_N)_{r,1}, \quad \mathcal{R}_{N,r}(x) = x^n E_{N,r}(x).$$

Proposition 2 says that $\mathcal{R}_{N,r}$ has a zero of order at least $2n$.

Choose $r_0 = 1/4$. On $|x| = r_0$, $|z| \leq 1/3$. The coefficients of y have modulus at most one, so

$$|y - 1| \leq \frac{1}{2}, \quad |\theta^j y| \leq \sum_{k \geq 1} k^3 (1/3)^k = \frac{33}{8} < 5 \quad (1 \leq j \leq 3).$$

Termwise estimates give $\|\mathcal{E}_0\|_\infty < 6000$.

For $m \geq 1$, put

$$D(m) = \text{diag}(1, m, m^2, m^3), \quad \mathcal{B}_m = D(m)^{-1} M_m D(m+1)/(m+1)^2.$$

On $|x| = 1/4$, direct estimates of the authoritative entries give

$$|(M_m)_{ij}| \leq 10^4 u^{i+2-j}, \quad u = 2m+3,$$

and therefore

$$|(\mathcal{B}_m)_{ij}| \leq 10^4 \left(\frac{u}{m}\right)^{i-1} \left(\frac{u}{m+1}\right)^{3-j}.$$

For $m \geq 1$, $u/m \leq 5$ and $u/(m+1) \leq 5/2$; summing four entries in each row gives the deliberately loose uniform bound

$$\|\mathcal{B}_m\|_\infty \leq 4 \cdot 10^8, \quad \|M_0\|_\infty < 10^7.$$

The balancing telescopes:

$$G_N = M_0(N!)^2 \mathcal{B}_1 \cdots \mathcal{B}_{N-1} D(N)^{-1}.$$

It follows that, for $N \geq 1$,

$$\max_{|x|=1/4} |\mathcal{R}_{N,r}(x)| \leq 6 \cdot 10^{10} (N!)^2 (4 \cdot 10^8)^{N-1} (1/4)^n. \quad (15)$$

Applying the maximum principle to $\mathcal{R}_{N,r}(x)/x^{2n}$ gives

$$|\mathcal{R}_{N,r}(x_0)| \leq 6 \cdot 10^{10} (N!)^2 (4 \cdot 10^8)^{N-1} (1/4)^n (4x_0)^{2n}. \quad (16)$$

Because $1 - z = 1/(1 - x)$,

$$Q_N(x_0) = (1 - x_0)^n \widehat{Q}_N(z_0).$$

Corollary 1 yields

$$Q_N(x_0) \geq 18 \cdot 29^N (N!)^2 (1 - x_0)^n.$$

Combining this with (16), we obtain

$$\left| \frac{E_{N,r}(x_0)}{q_N(x_0)} \right| = \left| \frac{\mathcal{R}_{N,r}(x_0)}{Q_N(x_0)} \right| \leq C(x_0) \beta(x_0)^N, \quad (17)$$

where $C(x_0) < \infty$ and

$$\beta(x_0) = \frac{4 \cdot 10^8}{29} \frac{x_0^2}{(1/4)(1-x_0)} = \frac{3125}{1307443596565949700399927} < 4 \cdot 10^{-19} < 1.$$

Therefore

$$\frac{E_{N,0}(x_0)}{q_N(x_0)} \rightarrow 0, \quad \frac{E_{N,1}(x_0)}{q_N(x_0)} \rightarrow 0. \quad (18)$$

8 Identification of the first-column limit

By (1) and the definition of Φ ,

$$A_0 - \Phi A_1 = -A(\mathcal{E}_0)_{0,*} - B(\mathcal{E}_0)_{1,*}.$$

Multiplying by $G_N \mathbf{e}_1$, dividing by $A_1 G_N \mathbf{e}_1 = S q_N$, and using (18), we get

$$\lim_{N \rightarrow \infty} \frac{A_0 G_N \mathbf{e}_1}{A_1 G_N \mathbf{e}_1} = \Phi(x_0).$$

Equation (3) therefore proves

$$\boxed{\lim_{N \rightarrow \infty} \frac{P_{N,1}}{Q_{N,1}} = \frac{\sqrt{10005}}{\pi}}. \quad (19)$$

9 The other three official columns

For completeness, we recall the exact finite-frame reduction already used to establish convergence of the recurrence. The balanced transfer tends to

$$\mathcal{S} = \begin{pmatrix} 64R - 44 & 96R - 54 & 48R - 17 & 8R \\ -8 & -12 & -6 & -1 \\ R^{-1} & -4R^{-1} & -6R^{-1} & -2R^{-1} \\ 2R^{-2} & (17R - 8)R^{-2} & 4(5R - 3)R^{-2} & (6R - 4)R^{-2} \end{pmatrix}.$$

Its characteristic polynomial is $Q_R(t)/R^2$, where

$$Q_R(t) = R^2 t^4 - (64R^3 - 56R^2 - 4)t^3 + (48R^2 - 262R + 220)t^2 - (12R - 8)t + 1.$$

The quartic is irreducible. Its spectral separation is also exact: on $|t| = 1$, the absolute value of its cubic coefficient exceeds the sum of the other coefficient magnitudes, because

$$(64R^3 - 56R^2 - 4) - (49R^2 - 250R + 213) = 64R^3 - 105R^2 + 250R - 217 > 0.$$

Rouché's theorem therefore places exactly three roots in $|t| < 1$ and the remaining root ρ in $|t| > 1$. Hence ρ is the unique root of maximal modulus.

We next remove any possible nonvanishing assumption about the denominator. The positivity estimate above and $Q_N = x_0^n q_N$ give

$$q_N(x_0) \geq 18 \cdot 29^N (N!)^2 \left(\frac{1-x_0}{x_0} \right)^{N+1}. \quad (20)$$

The scalar recurrence obtained from the first cyclic coordinate is of Poincaré type after the $(N!)^2$ balancing. The discrete Birkhoff–Poincaré theorem [4, Chapters 3 and 5] applies because the balanced coefficients are rational in N , have full expansions in N^{-1} , and the limiting spectrum is simple. If the coefficient of the ρ -mode in q_N were zero, the three-root separation just proved would give, for some $\tau < 1$,

$$|q_N(x_0)| \leq K_\tau (N!)^2 \tau^N.$$

This contradicts (20). Thus the dominant denominator coefficient is nonzero by a wholly exact argument.

It remains to transfer the first-column result to the other columns. For $r \geq 1$, put

$$\begin{aligned} F_r &= [\mathbf{e}_1, M_r \mathbf{e}_1, M_r M_{r+1} \mathbf{e}_1, M_r M_{r+1} M_{r+2} \mathbf{e}_1], \\ \gamma_{r,k} &= \prod_{\ell=1}^k (r+\ell)^2, \\ C_r &= [\mathbf{e}_1, \mathcal{B}_r \mathbf{e}_1, \mathcal{B}_r \mathcal{B}_{r+1} \mathbf{e}_1, \mathcal{B}_r \mathcal{B}_{r+1} \mathcal{B}_{r+2} \mathbf{e}_1]. \end{aligned}$$

The balancing telescopes exactly:

$$F_r = D(r) C_r \operatorname{diag}(\gamma_{r,0}, \dots, \gamma_{r,3}), \quad C_r \longrightarrow C = [\mathbf{e}_1, \mathcal{S} \mathbf{e}_1, \mathcal{S}^2 \mathbf{e}_1, \mathcal{S}^3 \mathbf{e}_1].$$

The limiting cyclic frame is nonsingular:

$$\det C = -\frac{4(27R-11)(128R^2-149R-43)}{R^6} \neq 0.$$

Thus F_r is invertible for all sufficiently large r . If $y_r(a) = a G_r \mathbf{e}_1$, exact inversion of this frame gives

$$a G_r \mathbf{e}_j = r^{-(j-1)} \sum_{k=0}^3 (C_r^{-1})_{k+1,j} \frac{y_{r+k}(a)}{\gamma_{r,k}}.$$

The same Birkhoff–Poincaré theorem supplies a linear dominant functional Λ and an exponent σ such that, for fixed k ,

$$\frac{y_{r+k}(a)}{(r!)^2 \rho^r r^\sigma \gamma_{r,k}} \longrightarrow \Lambda(a) \rho^k.$$

Consequently

$$\frac{a G_r \mathbf{e}_j}{(r!)^2 \rho^r r^{\sigma-(j-1)}} \longrightarrow \Lambda(a) \tilde{w}_j, \quad \tilde{w} = [1, \rho, \rho^2, \rho^3] C^{-1}.$$

An explicit left eigenvector is obtained from the first row of $R^2 \operatorname{adj}(tI - \mathcal{S})$. Each of its four coordinate polynomials is coprime to Q_R ; hence no coordinate vanishes at ρ . It is a nonzero multiple of $\tilde{w}$, so $\tilde{w}_j \neq 0$ for every j . Applying the last limit to $a = A_0, A_1$, using the exact denominator nonvanishing above, gives

$$\lim_{N \rightarrow \infty} \frac{P_{N,j}}{Q_{N,j}} = \frac{\Lambda(A_0)}{\Lambda(A_1)} \quad (j = 1, 2, 3, 4).$$

Equation (19) evaluates this common ratio. We conclude:

Theorem 1 (Ramanujan Challenge Problem 2.8). *For every official column $j = 1, 2, 3, 4$,*

$$\lim_{N \rightarrow \infty} \frac{P_{N,j}}{Q_{N,j}} = \frac{\sqrt{10005}}{\pi}.$$

Equivalently,

$$\lim_{N \rightarrow \infty} \frac{Q_{N,j}}{P_{N,j}} = \frac{\pi}{\sqrt{10005}}.$$

10 Reproducibility map

The proof package contains the following certificates.

File	Exact obligation
<code>p28_full_closure_certificate.wl</code>	Authoritative differential gauge; nonterminating tail contiguity; terminating adjoint equation; coefficientwise n -contiguity; normalization and top boundary; exact spectral and cyclic-frame closure.
<code>p28_kernel_contiguity_certificate.sage</code>	Independent coefficient/Ore proof of $M_N k_{N+1} = k_N$.
<code>p28_lattice_hypotheses_certificate.sage</code>	Rank-one factorization, tail direction, and transformed ODE identities.
<code>p28_convergence_constants.py</code>	Exact rational verification of the coefficient bounds, $\alpha_{n+1}/\alpha_n \geq 29n^2$, and $\beta(x_0) < 1$.
<code>all_four_columns_certificate.sage</code>	Balanced limit, Rouché separation, nonzero eigenvector coordinates, and invertible cyclic frame.
<code>p28_parametric_pade_probe.py</code>	Dependency-free finite exact regression of the predicted valuations.

The Wolfram certificate performs symbolic identities over $\mathbb{Q}(n, z)$; it uses no numerical samples. The Python constants check uses only the standard library’s `fractions.Fraction`. The SageMath files are independent exact cross-checks.

References

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