An Exact Hypergeometric Tail Certificate for Ramanujan Challenge Problem 2.8

Problem 2.8 submission

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Abstract

Let $G_N = M_0M_1 \cdots M_{N-1}$ be the 4×4 transfer product in Ramanujan Challenge Problem 2.8, and let $P_{N,j}$ and $Q_{N,j}$ be the two official seeded rows evaluated in column j. We prove

$$\lim_{N\to\infty} \frac{P_{N,j}}{Q_{N,j}} = \frac{\sqrt{10005}}{\pi} \qquad (j = 1, 2, 3, 4).$$

Equivalently, $Q_{N,j}/P_{N,j} \to \pi/\sqrt{10005}$.

The missing connection constant is fixed by an exact rank-three hypergeometric tail. A nonterminating ${}_4F_3$ Euler jet is carried backward by the displayed matrix, while the first denominator is a terminating adjoint ${}_4F_3$. Four denominator-cleared Ore identities and three coefficient identities give an all-N Padé divisibility theorem. Positivity at the negative CM point, a Cauchy estimate, and an explicit stable-graph contraction prove convergence for all four columns. Every algebraic identity used below is reproduced by a dependency-free rational-polynomial checker. The classical Chudnovsky formula, cited precisely in Section 2, is the sole imported theorem.

Contents

1 Statement and compact form of the seeds

Put

$$R = 151931373056001 = 53360^3 + 1, \qquad x_0 = \frac{1}{R}, \qquad z = -\frac{x}{1-x}.$$

Thus the official CM point is

$$z_0 = -\frac{1}{R-1} = -\frac{1}{53360^3}.$$

Let

$$G_N = M_0 M_1 \cdots M_{N-1}, \qquad G_0 = I_4,$$

where the rational family is defined as follows. Put $r = x^{-1}$, $\omega = u(3u-2)(3u+2)$, and

$$\begin{aligned}
a_1 &= r(144u^5 - 288u^4 + 144u^3) - 99u^5 + 333u^4 - 229u^3 - 114u^2 + 40u + 64,\\
a_2 &= r(432u^4 - 864u^3 + 432u^2) - 243u^4 + 909u^3 - 868u^2 - 80u + 272,\\
a_3 &= r(432u^3 - 864u^2 + 432u) - 153u^3 + 648u^2 - 860u + 360,\\
a_4 &= 144r(u-1)^2,\\
b_1 &= -144ru^3 + 9u^4 + 63u^3 + 158u^2 + 168u + 64,\\
b_2 &= 216ru^2 + 36u^3 - 189u^2 - 316u - 168,\\
b_3 &= 108ru + 54u^2 - 189u - 158,\\
c_1 &= -288r^2u^3 + r(54u^4 + 378u^3 + 948u^2 + 1008u + 384)\\
&\qquad + 18u^5 + 45u^4 - 251u^3 - 1086u^2 - 1384u - 576,\\
c_2 &= -432r^2u^2 + r(153u^4 - 657u^3 + 1292u^2 + 2064u + 1072)\\
&\qquad - 72u^4 + 702u^3 - 1069u^2 - 2508u - 1512,\\
c_3 &= -216r^2u + r(180u^3 - 891u^2 + 1450u + 1116)\\
&\qquad - 108u^3 + 864u^2 - 1385u - 1422,\\
c_4 &= -4r^2 + r(6u^2 - 33u + \tfrac{536}{9}) - 4u^2 + 32u - 63.
\end{aligned}$$

Define

$$\mathcal{M}(u,x) = \begin{pmatrix} a_1/\omega & a_2/\omega & a_3/\omega & a_4/\omega \\ -u^3 & -3u^2 & -3u & -1 \\ xb_1/144 & -xb_2/72 & -xb_3/36 & x(-2r-(2u-7))/2 \\ x^2c_1/288 & x^2c_2/144 & x^2c_3/72 & x^2c_4/4 \end{pmatrix}, \qquad M_N(x) = \mathcal{M}(2N+3, x). \tag{1}$$

At $x = x_0$, the only deformed challenge coefficient is restored by

$$236337691420383 = \frac{14R - 567}{9}.$$

Thus every use of Cauchy's theorem below concerns the explicitly displayed rational x-family, not an unspecified continuation.

Define four Pascal rows

$$b_0 = (1,0,0,0), \quad b_1 = (1,1,0,0),$$
$$b_2 = (1,2,1,0), \quad b_3 = (1,3,3,1)$$

and let $\mathcal{P}$ be the matrix with rows b_0, b_1, b_2, b_3. The compact denominator row is

$$C(x) = \left(\frac{18}{x} + \frac{159}{4},\ \frac{54}{x} + \frac{131}{2},\ \frac{54}{x} + 27,\ \frac{18}{x}\right).$$

Equivalently,

$$C = \frac{18}{x} b_3 + \frac{5}{4} b_0 + \frac{23}{2} b_1 + 27 b_2.$$

Set

$$A = 13591409, \qquad B = 545140134, \qquad S = 426880.$$

The two official initial rows have the exact form

$$A_1 = SC, \qquad A_0 = AC - \frac{5}{4} H_0, \qquad H_0 = A b_0 + B b_1 = (A + B, B, 0, 0). \tag{2}$$

At $x = x_0$, these identities reproduce the official integer rows entry by entry:

$$\begin{aligned} A_0 = (&37169305760442252761441, 111507917281327441564208,\\ &111507917281327599720129, 37169305760442410917362),\\ A_1 = (&11674163615426396923 20, 35022490846278961321 60,\\ &35022490846278796972 80, 11674163615426227238 40). \end{aligned}$$

For $j = 1, \ldots, 4$, write

$$P_{N,j} = A_0 G_N \mathbf{e}_j, \qquad Q_{N,j} = A_1 G_N \mathbf{e}_j.$$

We first identify the first-column limit and then invoke the exact cyclic frame to cover all four columns.

2 The CM function and its exact value

Let

$$y(z) = {}_3F_2\left(\begin{matrix} \frac{1}{6}, \frac{1}{2}, \frac{5}{6} \\ 1, 1 \end{matrix}; z\right), \qquad \theta = z\frac{d}{dz} = (1 - x)x\frac{d}{dx},$$

and define

$$\Phi(x) = \frac{A y(z) + B \theta y(z)}{S}. \tag{3}$$

We use one external theorem: the classical Chudnovsky identity. In the normalization used here it is Theorem 0.1 of Milla's equation-by-equation derivation [2], whose modular and CM proof is completed in Theorem 9.7 and Chapter 10:

$$\frac{1}{\pi} = \frac{12}{640320^{3/2}} \sum_{k=0}^{\infty} \frac{(6k)!}{(3k)!(k!)^3} (A + Bk)(-640320^{-3})^k.$$

The elementary coefficient identity

$$\frac{(6k)!}{(3k)!(k!)^3} = 1728^k \frac{(\frac{1}{6})_k (\frac{1}{2})_k (\frac{5}{6})_k}{(k!)^3}$$

and $640320 = 12 \cdot 53360$ give

$$A y(z_0) + B \theta y(z_0) = \frac{640320^{3/2}}{12\pi} = \frac{426880\sqrt{10005}}{\pi}.$$

Consequently,

$$\boxed{\Phi(x_0) = \frac{\sqrt{10005}}{\pi}.} \tag{4}$$

3 The nonterminating adjoint tail

Put $n = N + 1$ and $\delta_N = \theta - n$. Define

$$F_N(z) = \kappa_N z^n {}_4F_3\left(\begin{matrix} n, n+\frac{1}{6}, n+\frac{1}{2}, n+\frac{5}{6} \\ 2n, 2n, 2n \end{matrix}; z\right), \tag{5}$$

where

$$\kappa_0 = \frac{5}{72}, \qquad \frac{\kappa_{N+1}}{\kappa_N} = -\frac{(6N+7)(6N+11)}{576(N+1)^2(2N+3)^2}.$$

Its Euler jet is

$$k_N = \left(F_N, \delta_N F_N, \delta_N^2 F_N, \delta_N^3 F_N\right)^T.$$

Proposition 1 (Exact tail contiguity)**.** *For every $N \geq 0$,*

$$\boxed{M_N k_{N+1} = k_N.} \tag{6}$$

Proof. Put $m = (u-1)/2 = N + 1$. For the r-th row of $\mathcal{M}(u, x)$, define

$$P_r(t) = \sum_{s=0}^{3} \mathcal{M}(u, x)_{r+1,s+1} t^s, \qquad \mathcal{T}Q = (1-x)x\partial_x Q + (t+1)Q.$$

The shifted tail satisfies

$$L_+(t) F_{N+1} = 0,$$

where

$$L_+(t) = (1-x)t(t+u)^3 + x(t+m+1)(t+m+\tfrac{7}{6})(t+m+\tfrac{3}{2})(t+m+\tfrac{11}{6}).$$

There is no division step: direct expansion gives the following four denominator-cleared polynomial identities:

$$u(3u-2)(3u+2)x(\mathcal{T}P_0 - P_1) = 144(u-1)^2 L_+, \tag{7}$$

$$\mathcal{T}P_1 - P_2 = -L_+, \tag{8}$$

$$2(\mathcal{T}P_2 - P_3) = (-2 + 7x - 2ux)L_+, \tag{9}$$

$$36(\mathcal{T}P_3 + \ell_3 P_3 + \ell_2 P_2 + \ell_1 P_1 + \ell_0 P_0) = q_3 L_+, \tag{10}$$

with

$$\begin{aligned}
\ell_0 &= (u-1)u(3u-2)(3u+2)x/144, \\
\ell_1 &= [-576 + 864u - 432u^2 + 72u^3 + (580 - 872u + 405u^2 - 36u^3)x]/72, \\
\ell_2 &= [432 - 432u + 108u^2 + (-436 + 405u - 54u^2)x]/36, \\
\ell_3 &= (-12 + 6u + 11x - 2ux)/2, \\
q_3 &= -36 + (536 - 297u + 54u^2)x + (-567 + 288u - 36u^2)x^2.
\end{aligned}$$

For clarity, the coefficient calculation producing the first row is also written without a special-function routine. Set

$$A(t) = \frac{144(u-1)^2(t+u)^3}{u(3u-2)(3u+2)}, \qquad B(t) = P_0(t) - \frac{A(t)}{x},$$

$$\varrho = -\frac{(3u-2)(3u+2)}{144(u-1)^2u^2},$$

$$\chi_j = \frac{(m+j)(m+\frac{1}{6}+j)(m+\frac{1}{2}+j)(m+\frac{5}{6}+j)}{m(m+\frac{1}{6})(m+\frac{1}{2})(m+\frac{5}{6})} \times \left(\frac{2m(2m+1)}{(2m+j)(2m+j+1)}\right)^3,$$

$$\psi_j = \frac{(m+j)(m+\frac{1}{6}+j)(m+\frac{1}{2}+j)(m+\frac{5}{6}+j)}{(2m+j)^3(j+1)}.$$

The constant and generic coefficient equations are

$$\varrho[-A(0)] = 1, \qquad \varrho\left[-A(j)\chi_j + \frac{(A(j-1)+B(j-1))\chi_{j-1}}{\psi_{j-1}}\right] = 1 \quad (j \geq 1). \tag{11}$$

Clearing the displayed nonzero factors turns (7)–(11) into polynomial equalities with every coefficient zero. The independent script `p28_standalone_equations.py` expands precisely these equalities using only rational addition and multiplication. Equation (11) gives the first row of (6); applying (7)–(10) successively gives the other three Euler-jet rows. □

For $N = 0$, write $y(z) = 1 + \sum_{k\geq 1} c_k z^k$. The coefficient of z is

$$c_1 = \frac{(1/6)(1/2)(5/6)}{1^3} = \frac{5}{72},$$

and, after shifting $k \mapsto k+1$, both sides below have initial coefficient 5/72 and consecutive-coefficient ratio

$$\frac{(k+\frac{1}{6})(k+\frac{1}{2})(k+\frac{5}{6})}{(k+1)^3}.$$

Hence coefficient equality, rather than a named ascension rule, gives

$$F_0 = y - 1 = \frac{5}{72} z\, {}_4F_3\left(\begin{matrix}1, \frac{7}{6}, \frac{3}{2}, \frac{11}{6} \\ 2, 2, 2\end{matrix}; z\right). \tag{12}$$

Because $\mathcal{P}$ is the Pascal matrix,

$$\mathcal{P}k_0 = (y-1, \theta y, \theta^2 y, \theta^3 y)^T.$$

4 The rank-three error carrier

Set

$$f = (y-1, \theta y, \theta^2 y, \theta^3 y)^T, \qquad \mathcal{E}_0 = \frac{5}{4}\mathcal{P} + fC, \qquad \mathcal{E}_N = \mathcal{E}_0 G_N.$$

The transformed hypergeometric equation is

$$72\theta^3 y + 108x\theta^2 y + 46x\theta y + 5xy = 0. \tag{13}$$

Indeed the coefficient ratio of y gives

$$\left[\theta^3 - z(\theta + \tfrac{1}{6})(\theta + \tfrac{1}{2})(\theta + \tfrac{5}{6})\right] y = 0.$$

Substituting $z = -x/(1-x)$, using $\theta = (1-x)x\partial_x$, and multiplying by $72(1-x)$ expands to (13). Using the displayed decomposition of C, equations (12)–(13) give

$$Ck_0 = -\frac{5}{4}.$$

It follows that

$$\mathcal{E}_0 k_0 = \frac{5}{4} f + f(Ck_0) = 0.$$

Proposition 1 therefore implies

$$\boxed{\mathcal{E}_N k_N = 0 \qquad (N \geq 0).} \tag{14}$$

The same differential equation gives the exact row relation

$$72(\mathcal{E}_N)_{3,*} + 108x(\mathcal{E}_N)_{2,*} + 46x(\mathcal{E}_N)_{1,*} + 5x(\mathcal{E}_N)_{0,*} = 0. \tag{15}$$

5 A discrete valuation lemma

Let

$$H = \mathrm{diag}(x, 1, 1, 1), \qquad J_N = HM_N.$$

Every entry of J_N is regular at $x = 0$. If $u = 2N + 3$ and

$$a_N = \frac{144(u-1)^2}{u(3u-2)(3u+2)}, \qquad V_N = (u^3, 3u^2, 3u, 1),$$

direct substitution in the authoritative matrix gives

$$J_N(0) = \begin{pmatrix} a_N \\ -1 \\ -1 \\ -1 \end{pmatrix} V_N. \tag{16}$$

Thus $J_N(0)$ has rank one.

The first nonconstant coefficient of the ${}_4F_3$ in (5) equals

$$c_N = \frac{u(3u-2)(3u+2)}{144(u-1)^2} = a_N^{-1}.$$

Since $z = -x + O(x^2)$,

$$Hk_N = x^{N+2}\eta_N \left[\begin{pmatrix} 1 \\ -c_N \\ -c_N \\ -c_N \end{pmatrix} + O(x) \right], \qquad \eta_N \neq 0. \tag{17}$$

The leading vector in (17) is precisely the image direction in (16). The other expansion required below is equally direct. Since

$$F_{N+1} = \kappa_{N+1} z^{N+2}(1 + O(z)), \qquad \delta_{N+1} = \theta - (N+2),$$

the shifted derivatives kill the leading monomial, and therefore

$$k_{N+1} = x^{N+2}\widetilde{\eta}_N \left(\mathbf{e}_1 + xs_N + O(x^2)\right), \qquad \widetilde{\eta}_N \neq 0. \tag{18}$$

Moreover, (16) gives

$$J_N(0)\mathbf{e}_1 = u^3(a_N, -1, -1, -1)^T \neq 0.$$

Lemma 1 (DVR step, including the extra first-column zero)**.** *Let $R_0 = \mathbb{Q}[[x]]$, $H = \operatorname{diag}(x, 1, 1, 1)$, and suppose*

$$E = x^N LH, \qquad L \in \operatorname{Mat}_4(R_0), \qquad Ek = 0.$$

Assume $J = HM \in \operatorname{Mat}_4(R_0)$, $Mk^+ = k$, and

$$k^+ = x^r\alpha(\mathbf{e}_1 + xs + O(x^2)),$$
$$Hk = x^r\beta(v_0 + xv_1 + O(x^2)),$$

with $\alpha\beta \neq 0$. If $J(0)$ has rank one, $\operatorname{im} J(0) = \mathbb{Q}v_0$, and $J(0)\mathbf{e}_1 \neq 0$, then

$$EM = x^{N+1}L^+H$$

for some $L^+ \in \operatorname{Mat}_4(R_0)$.

Proof. Absorb β/α into v_0, v_1. From $Jk^+ = Hk$,

$$J(\mathbf{e}_1 + xs + O(x^2)) = v_0 + xv_1 + O(x^2).$$

Write $L = L_0 + xL_1 + \cdots$. The equation $L(Hk) = 0$ gives

$$L_0v_0 = 0, \qquad L_0v_1 + L_1v_0 = 0.$$

Since $J(0)$ has image $\mathbb{Q}v_0$, $L_0J(0) = 0$, so LJ is entrywise divisible by x. The coefficient of x in its first column is

$$L_0(v_1 - J(0)s) + L_1v_0 = -L_0J(0)s = 0.$$

Hence that column is divisible by x^2. Therefore

$$L^+ = x^{-1}LJH^{-1}$$

is regular and $EM = x^{N+1}L^+H$. □

Proposition 2 (All-N Padé divisibility)**.** *For every $N \geq 0$, there is $L_N \in \operatorname{Mat}_4(\mathbb{Q}[[x]])$ such that*

$$\boxed{\mathcal{E}_N = x^N L_N H.} \tag{19}$$

Thus every row of $\mathcal{E}_N$ has componentwise valuations at least

$$(N + 1, N, N, N).$$

The last row has the stronger valuations

$$(N + 2, N + 1, N + 1, N + 1).$$

Proof. Every component of f is $O(x)$, while C has only a simple pole. All four components of f have leading term $-5x/72$. Consequently the constant term in the first column of fC is $-5/4$, cancelling the first component of every row of $(5/4)\mathcal{P}$. Hence $\mathcal{E}_0 = L_0H$.

Apply Lemma 1 inductively, using (14), (16), (17), (18), the displayed nonzero first column of $J_N(0)$, and $M_Nk_{N+1} = k_N$. The stronger last-row assertion follows from (15). □

6 The terminating denominator

For the first column put

$$q_N(x) = CG_N\mathbf{e}_1, \qquad n = N+1, \qquad Q_N(x) = x^n q_N(x),$$

and define

$$\widehat{Q}_N(z) = (1-z)^n Q_N\left(-\frac{z}{1-z}\right).$$

Since $x = -z/(1-z)$, this is also the first component of $(-z)^n CG_N$.

Proposition 3 (Exact terminating denominator)**.** *For every $N \geq 0$,*

$$\frac{\widehat{Q}_N(z)}{\alpha_n} = {}_4F_3\left(\begin{matrix}-n, -n-\frac{1}{6}, -n-\frac{1}{2}, -n-\frac{5}{6} \\ 1-2n, 1-2n, 1-2n\end{matrix}; z\right), \tag{20}$$

where

$$\alpha_1 = 18, \qquad \frac{\alpha_{n+1}}{\alpha_n} = \frac{576n^2(2n+1)^2}{(6n+1)(6n+5)}. \tag{21}$$

Here and below the hypergeometric expression denotes the unambiguous finite sum over $0 \leq k \leq n$; it terminates before any lower Pochhammer symbol can vanish.

Proof. Write $p_n = \widehat{Q}_{n-1}$ for $n \geq 1$. We use coefficient induction, because normalization at $z = 0$ alone would not select a unique solution of the fourth-order equation. To expose the matrix-to-scalar bridge, write

$$\mathcal{L}_n(t) = t(t+2n-1)^3 - z(t+n)(t+n+\tfrac{1}{6})(t+n+\tfrac{1}{2})(t+n+\tfrac{5}{6}) = \sum_{j=0}^{4} c_j t^j$$

and define, for operator polynomials with coefficients on the left,

$$\Theta Q = z\partial_z Q + tQ.$$

The four horizontal components are reconstructed from the first by

$$\pi_0 = 1, \qquad \pi_3 = \frac{c_4}{c_0}t, \qquad \pi_2 = \frac{c_3}{c_4}\pi_3 - \Theta\pi_3, \qquad \pi_1 = \frac{c_2}{c_4}\pi_3 - \Theta\pi_2.$$

The remaining horizontal equation is the direct factorization

$$\begin{aligned}\Theta\pi_1 + 1 - \frac{c_1}{c_4}\pi_3 = {} & -\frac{72}{n(2n+1)(6n+1)(6n+5)z}\big[t(t-2n)^3 \\ & \qquad - z(t-n)(t-n-\tfrac{1}{6})(t-n-\tfrac{1}{2})(t-n-\tfrac{5}{6})\big].\end{aligned}$$

Let

$$\mathcal{C}_n(z) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -c_0/c_4 & -c_1/c_4 & -c_2/c_4 & -c_3/c_4 \end{pmatrix}.$$

The propagation of the reconstructed row is the following sixteen-entry identity:

$$\mathcal{C}_n[-z\mathcal{M}(2n+1, -z/(1-z))] - \theta[-z\mathcal{M}(2n+1, -z/(1-z))] - [-z\mathcal{M}(2n+1, -z/(1-z))]\mathcal{C}_{n+1} = 0. \tag{22}$$

Finally, direct contraction with the displayed matrix gives

$$\sum_{r=0}^{3} \pi_r(t)\,[-z\mathcal{M}(2n+1,-z/(1-z))]_{r+1,1} = d_0(t) + zd_1(t). \tag{23}$$

These are identities in $\mathbb{Q}(n,z)(t)$. The factors depending on the integer n are nonzero for $n \geq 1$; no value is obtained by dividing at $z=0$, because the verifier cross-multiplies first and the reconstructed base functions have removable limits there. The mandatory sparse-polynomial verifier checks every reconstruction equation, the closing factorization, and (22)–(23) by cross multiplication. For any reconstructed horizontal row these identities give

$$p_{n+1} = (d_0(\theta) + zd_1(\theta))p_n, \tag{24}$$

where

$$d_0(t) = \frac{72(2n+1)^2(2n-t)^3}{n(6n+1)(6n+5)}$$

and

$$d_1(t) = -\frac{P(n,t)}{n(2n+1)(6n+1)(6n+5)}$$

with

$$\begin{aligned} P(n,t) = &-5n - 76n^2 + 1404n^3 + 4360n^4 + 4320n^5 + 1440n^6 \\ &+ (5 + 127n - 1760n^2 - 6536n^3 - 7632n^4 - 3024n^5)t \\ &+ (-51 + 659n + 3086n^2 + 4500n^3 + 2232n^4)t^2 \\ &+ (-72 - 432n - 864n^2 - 576n^3)t^3. \end{aligned}$$

Thus (24) is the scalar form of the displayed matrix recurrence; no differential-equation uniqueness is used below.

Let

$$h_{n,k} = \frac{(-n)_k(-n-\frac{1}{6})_k(-n-\frac{1}{2})_k(-n-\frac{5}{6})_k}{(1-2n)_k^3 k!} \quad (0 \leq k \leq n), \qquad \nu_n = \frac{576n^2(2n+1)^2}{(6n+1)(6n+5)}.$$

For $n \geq 1$, clearing the displayed nonzero denominators gives exactly

$$d_0(0) = \nu_n, \tag{25}$$

$$d_0(k) + d_1(k-1)\frac{h_{n,k-1}}{h_{n,k}} = \nu_n \frac{h_{n+1,k}}{h_{n,k}} \qquad (1 \leq k \leq n), \tag{26}$$

$$d_1(n) = -\nu_n \frac{(n+\frac{7}{6})(n+\frac{3}{2})(n+\frac{11}{6})}{8(2n+1)^3}. \tag{27}$$

The base row is not assumed: from the first component of C,

$$q_0(x) = \frac{18}{x} + \frac{159}{4}, \qquad Q_0(x) = xq_0(x) = 18 + \frac{159}{4}x,$$

and therefore

$$p_1(z) = (1-z)Q_0\left(-\frac{z}{1-z}\right) = 18\left(1 - \frac{77}{24}z\right).$$

Direct application of the four displayed π_r operators gives the four base identities

$$-zC\left(-\frac{z}{1-z}\right) = (\pi_0(\theta)p_1, \pi_1(\theta)p_1, \pi_2(\theta)p_1, \pi_3(\theta)p_1)|_{n=1}\,. \tag{28}$$

The fourth base equation is also direct:

$$\left[\theta(\theta-2)^3 - z(\theta-1)(\theta-\tfrac{7}{6})(\theta-\tfrac{3}{2})(\theta-\tfrac{11}{6})\right]p_1 = 0.$$

Equivalently, if $H_1 = -zC(-z/(1-z))$, then all four components of

$$\theta H_1 + H_1\mathcal{C}_1 = 0$$

vanish. The mandatory verifier checks both forms independently. Moreover,

$$\deg((d_0(\theta) + zd_1(\theta))p_n) \le n+1.$$

Equations (25)–(27) therefore prove, coefficient by coefficient, that the right side of (24) is $\alpha_{n+1}\sum_{k=0}^{n+1} h_{n+1,k}z^k$. Starting from (28), (22) propagates the horizontal form at every step. This proves (20) and (21) for the actual row CG_N, not merely for a scalar surrogate. The mandatory standalone checker independently expands the cleared identities and rejects any nonzero coefficient. $\square$

Corollary 1 (Positivity at the CM point)**.** *At $z_0 = -1/53360^3$,*

$$\widehat{Q}_N(z_0) \ge \alpha_n > 0.$$

Moreover,

$$\alpha_n \ge 18 \cdot 29^N (N!)^2.$$

Proof. For $0 \le k \le n$, the coefficient of z^k in (20) has sign $(-1)^k$. Since $z_0 < 0$, every summand is nonnegative. Also

$$\frac{576n^2(2n+1)^2}{(6n+1)(6n+5)} - 29n^2 = \frac{n^2(1260n^2+1260n+431)}{(6n+1)(6n+5)} > 0.$$

Iterating (21) proves the lower bound. $\square$

7 From formal contact to convergence at x_0

This step is included to rule out a beyond-all-orders ambiguity. For $r = 0, 1$, let

$$E_{N,r}(x) = (\mathcal{E}_N)_{r,1}, \qquad \mathcal{R}_{N,r}(x) = x^n E_{N,r}(x).$$

Proposition 2 says that $\mathcal{R}_{N,r}$ has a zero of order at least $2n$.

Choose $r_0 = 1/4$. On $|x| = r_0$, $|z| \le 1/3$. The coefficients of y have modulus at most one, so

$$|y-1| \le \frac{1}{2}, \qquad |\theta^j y| \le \sum_{k\ge1} k^3(1/3)^k = \frac{33}{8} < 5 \quad (1 \le j \le 3).$$

Termwise estimates give $\|\mathcal{E}_0\|_\infty < 6000$.

For $m \geq 1$, put

$$D(m) = \mathrm{diag}(1, m, m^2, m^3), \qquad \mathcal{B}_m = D(m)^{-1} M_m D(m+1)/(m+1)^2.$$

On $|x| = 1/4$, direct estimates of the authoritative entries give

$$|(M_m)_{ij}| \leq 10^4 u^{i+2-j}, \qquad u = 2m+3,$$

and therefore

$$|(\mathcal{B}_m)_{ij}| \leq 10^4 \left(\frac{u}{m}\right)^{i-1} \left(\frac{u}{m+1}\right)^{3-j}.$$

For $m \geq 1$, $u/m \leq 5$ and $u/(m+1) \leq 5/2$. Because the exponent $3-j$ is negative when $j = 4$, we use the exact case split

$$\left(\frac{u}{m+1}\right)^{3-j} \leq \begin{cases} (5/2)^{3-j} \leq 25/4, & j \leq 3, \\ 1/2, & j = 4, \end{cases}$$

where the second line follows from $u/(m+1) = 2 + 1/(m+1) \geq 2$. Thus every entry is at most 7,812,500, and summing each row gives

$$\|\mathcal{B}_m\|_\infty \leq 31{,}250{,}000 < 4 \cdot 10^8, \qquad \|M_0\|_\infty < 10^7.$$

The balancing telescopes:

$$G_N = M_0 (N!)^2 \mathcal{B}_1 \cdots \mathcal{B}_{N-1} D(N)^{-1}.$$

It follows that, for $N \geq 1$,

$$\max_{|x|=1/4} |\mathcal{R}_{N,r}(x)| \leq 6 \cdot 10^{10} (N!)^2 (4 \cdot 10^8)^{N-1} (1/4)^n. \tag{29}$$

We now verify the analytic hypothesis behind the next step. On $|x| \leq 1/4$,

$$\left| -\frac{x}{1-x} \right| \leq \frac{1}{3} < 1,$$

so the series defining y and its first three Euler derivatives are holomorphic on a neighborhood of the closed disk. Because $f = O(x)$, the simple pole of C cancels in fC, so $\mathcal{E}_0$ is holomorphic there. Inspection of (1) shows that each M_m is holomorphic off $x = 0$ in this disk and has at most a simple pole at 0. Hence $\mathcal{R}_{N,r} = x^n (\mathcal{E}_N)_{r,1}$, with $n = N+1$, is holomorphic on the disk after removing its possible singularity at zero. Proposition 2 strengthens its order there to at least $2n$. Therefore $\mathcal{R}_{N,r}(x)/x^{2n}$ has a removable singularity at zero and is holomorphic on a neighborhood of the closed disk. Applying the maximum principle to $\mathcal{R}_{N,r}(x)/x^{2n}$ gives

$$|\mathcal{R}_{N,r}(x_0)| \leq 6 \cdot 10^{10} (N!)^2 (4 \cdot 10^8)^{N-1} (1/4)^n (4x_0)^{2n}. \tag{30}$$

Because $1 - z = 1/(1-x)$,

$$Q_N(x_0) = (1-x_0)^n \widehat{Q}_N(z_0).$$

Corollary 1 yields

$$Q_N(x_0) \geq 18 \cdot 29^N (N!)^2 (1-x_0)^n.$$

Combining this with (30), we obtain

$$\left| \frac{E_{N,r}(x_0)}{q_N(x_0)} \right| = \left| \frac{\mathcal{R}_{N,r}(x_0)}{Q_N(x_0)} \right| \leq K_0\, \beta(x_0)^N, \tag{31}$$

where $K_0 < \infty$ is independent of N and

$$\beta(x_0) = \frac{4 \cdot 10^8}{29} \frac{x_0^2}{(1/4)(1-x_0)} = \frac{3125}{130744359656594970039 9927} < 4 \cdot 10^{-19} < 1.$$

Therefore

$$\frac{E_{N,0}(x_0)}{q_N(x_0)} \longrightarrow 0, \qquad \frac{E_{N,1}(x_0)}{q_N(x_0)} \longrightarrow 0. \tag{32}$$

8 Identification of the first-column limit

By (2) and the definition of Φ,

$$A_0 - \Phi A_1 = -A(\mathcal{E}_0)_{0,*} - B(\mathcal{E}_0)_{1,*}.$$

Multiplying by $G_N \mathbf{e}_1$, dividing by $A_1 G_N \mathbf{e}_1 = S q_N$, and using (32), we get

$$\lim_{N\to\infty} \frac{A_0 G_N \mathbf{e}_1}{A_1 G_N \mathbf{e}_1} = \Phi(x_0).$$

Equation (4) therefore proves

$$\boxed{\lim_{N\to\infty} \frac{P_{N,1}}{Q_{N,1}} = \frac{\sqrt{10005}}{\pi}.} \tag{33}$$

9 The other three official columns

All matrices in this section are evaluated at $x = x_0 = 1/R$. For $m \geq 1$, retain

$$D_m = \mathrm{diag}(1, m, m^2, m^3), \qquad \mathcal{B}_m = D_m^{-1} M_m D_{m+1}/(m+1)^2,$$

and, for any row a, put

$$Z_m(a) = \frac{a G_m D_m}{(m!)^2}.$$

Then the balancing gives the exact recurrence

$$Z_{m+1}(a) = Z_m(a)\mathcal{B}_m. \tag{34}$$

Substitution in the displayed matrix shows, entry by entry, that $\mathcal{B}_m - \mathcal{S} = O(m^{-1})$, where

$$\mathcal{S} = \begin{pmatrix} 64R-44 & 96R-54 & 48R-17 & 8R \\ -8 & -12 & -6 & -1 \\ R^{-1} & -4R^{-1} & -6R^{-1} & -2R^{-1} \\ 2R^{-2} & (17R-8)R^{-2} & 4(5R-3)R^{-2} & (6R-4)R^{-2} \end{pmatrix}.$$

Its characteristic polynomial is $Q_R(t)/R^2$, where

$$\begin{aligned} Q_R(t) = {} & R^2 t^4 - (64R^3 - 56R^2 - 4)t^3 \\ & + (48R^2 - 262R + 220)t^2 - (12R - 8)t + 1. \end{aligned}$$

On $|t| = 1$, the cubic coefficient strictly dominates the sum of the other four coefficient magnitudes, because

$$(64R^3 - 56R^2 - 4) - (49R^2 - 250R + 213) = 64R^3 - 105R^2 + 250R - 217 > 0.$$

To count the roots without an irreducibility or root-finder call, consider

$$H_s(t) = -(64R^3 - 56R^2 - 4)t^3 + s\{R^2t^4 + (48R^2 - 262R + 220)t^2 - (12R - 8)t + 1\}.$$

The strict inequality above gives $H_s(t) \neq 0$ for $|t| = 1$, $0 \leq s \leq 1$. Thus the winding number of $H_s(e^{i\vartheta})$ about zero cannot change with s; at $s = 0$ it is 3. Hence $Q_R = H_1$ has three zeros in $|t| < 1$ and one, counted with multiplicity, in $|t| > 1$. Moreover

$$Q_R(1) = -(64R^3 - 105R^2 + 274R - 233) < 0, \qquad \lim_{t\to+\infty} Q_R(t) = +\infty.$$

Consequently the unique exterior zero is a simple real number $\rho > 1$.

Lemma 2 (Explicit dominant-product dichotomy)**.** *Fix τ with*

$$\max_{\lambda\neq\rho} |\lambda| < \tau < 1,$$

Then there exist a late index m_0, a linear functional Λ on rows, nonzero scalars L_m independent of the row, and a left ρ-eigenvector w of $\mathcal{S}$ such that the following alternatives hold:

$$\Lambda(a) = 0 \implies \|Z_m(a)\| \leq K_a\tau^{m-m_0}, \tag{35}$$
$$\Lambda(a) \neq 0 \implies Z_m(a) = \Lambda(a)L_m(w + o_a(1)). \tag{36}$$

Proof. Choose an invertible P with

$$P^{-1}\mathcal{S}P = \begin{pmatrix} \rho & 0 \\ 0 & A \end{pmatrix}, \qquad \operatorname{spr}(A) < 1.$$

Choose $\operatorname{spr}(A) < \theta < \tau$. For stable rows define

$$\|\beta\|_\theta = \sum_{k=0}^{\infty} \theta^{-k}\|\beta A^k\|_0.$$

The finite Jordan identity

$$J_\lambda^k = \sum_{\ell=0}^{s-1} \binom{k}{\ell} \lambda^{k-\ell}N^\ell$$

gives, for $\operatorname{spr}(A) < \eta < \theta$, $\|A^k\|_0 \leq Ck^2\eta^k$; hence the series converges and

$$\|\beta A\|_\theta = \theta \sum_{k=1}^{\infty} \theta^{-k}\|\beta A^k\|_0 \leq \theta\|\beta\|_\theta.$$

Use the dual norm for stable columns.

Write

$$T_m = P^{-1}\mathcal{B}_mP = \begin{pmatrix} a_m & b_m \\ c_m & E_m \end{pmatrix}.$$

Then $a_m \to \rho$, $b_m, c_m \to 0$, and $E_m \to A$. Choose

$$\theta < d_* < \tau < 1 < a_* < \rho.$$

For sufficiently large m_0, a positive ϵ makes, for $m \geq m_0$,

$$|a_m| \geq a_*, \quad \|E_m\| \leq d_*, \quad \|b_m\|, \|c_m\| \leq \epsilon, \tag{37}$$

$$d_* + \epsilon < \tau, \quad a_* - \epsilon > 1, \quad d_* + \epsilon < a_* - \epsilon, \tag{38}$$

$$q := \frac{d_*}{a_* - \epsilon} + \frac{(d_* + \epsilon)\epsilon}{(a_* - \epsilon)^2} < 1. \tag{39}$$

For stable columns $\|h\| \leq 1$, set

$$\Psi_m(h) = \frac{E_m h - c_m}{a_m - b_m h}.$$

Equations (37)–(38) give

$$\|\Psi_m(h)\| \leq \frac{d_* + \epsilon}{a_* - \epsilon} < 1.$$

For two such columns,

$$\Psi_m(h) - \Psi_m(k) = \frac{E_m(h-k)}{a_m - b_m h} + \frac{(E_m k - c_m) b_m (h-k)}{(a_m - b_m h)(a_m - b_m k)},$$

so (39) gives

$$\|\Psi_m(h) - \Psi_m(k)\| \leq q\|h - k\|.$$

For $M > m$, set $h_M^{(M)} = 0$ and recurse backward by $h_j^{(M)} = \Psi_j(h_{j+1}^{(M)})$. Then, for $M' > M$,

$$\|h_m^{(M')} - h_m^{(M)}\| \leq 2q^{M-m}.$$

Thus $h_m = \lim_{M\to\infty} h_m^{(M)}$ exists and satisfies

$$h_m a_m + c_m = (h_m b_m + E_m) h_{m+1}. \tag{40}$$

The defining recurrence gives the explicit bound

$$\|h_m\| \leq \frac{\|E_m\|\|h_{m+1}\| + \|c_m\|}{|a_m| - \|b_m\|}.$$

Together with $\|A\|_\theta/\rho < 1$, this gives

$$\limsup_{m\to\infty} \|h_m\| \leq \frac{\theta}{\rho} \limsup_{m\to\infty} \|h_m\|, \qquad \text{hence} \qquad h_m \to 0.$$

Write

$$U_m(a) = Z_m(a)P = (\alpha_m, \beta_m), \qquad \xi_m = \alpha_m - \beta_m h_m, \qquad d_m = a_m - b_m h_{m+1}.$$

Using (40) in $U_{m+1} = U_m T_m$ gives the exact scalar equation

$$\xi_{m+1} = d_m \xi_m.$$

Define the composed seed functional and product

$$\Lambda(a) = \alpha_{m_0}(a) - \beta_{m_0}(a) h_{m_0}, \qquad L_m = \prod_{\ell=m_0}^{m-1} d_\ell.$$

Both are now explicit, Λ is linear, and $\xi_m = \Lambda(a)L_m$. Equations (37)–(38) ensure $d_m \neq 0$.

If $\Lambda(a) = 0$, then $\alpha_m = \beta_m h_m$ and

$$\beta_{m+1} = \beta_m(E_m + h_m b_m), \qquad \|\beta_{m+1}\| < (d_* + \epsilon)\|\beta_m\| < \tau\|\beta_m\|,$$

which proves (35).

If $\Lambda(a) \neq 0$, put $r_m = \beta_m/\xi_m$. Exact substitution gives

$$r_{m+1} = \frac{b_m + r_m(E_m + h_m b_m)}{d_m}.$$

Here $b_m/d_m \to 0$ and $(E_m + h_m b_m)/d_m \to A/\rho$, whose norm is below one. Enlarge m_0 once more, redefining Λ and L_m from this new index, so that for some $q_1 < 1$,

$$\left\|\frac{E_m + h_m b_m}{d_m}\right\| \leq q_1 \qquad (m \geq m_0).$$

Then, for every $m \geq m_0$,

$$\|r_m\| \leq q_1^{m-m_0}\|r_{m_0}\| + \sum_{\ell=m_0}^{m-1} q_1^{m-1-\ell}\left\|\frac{b_\ell}{d_\ell}\right\| \longrightarrow 0.$$

Since $\alpha_m/\xi_m = 1 + r_m h_m \to 1$,

$$U_m(a) = \Lambda(a)L_m\big((1,0) + o_a(1)\big).$$

Multiplying by P^{-1} proves (36) with

$$w = (1,0)P^{-1}, \qquad w\mathcal{S} = \rho w.$$

$\square$

No GCD or irreducibility decision is needed to show that all four coordinates of w are nonzero. Define the polynomial row $w(t)$ by

$$\begin{aligned}
w_1(t) &= Rt(Rt^2 - 7) + 12R^2t^2 + 4t^2 + 44t + 10,\\
\frac{w_2(t)}{2} &= R^2t((48R - 27)t - 28) + (194R - 108)t + 40R - 23,\\
w_3(t) &= R^2t((48R - 17)t - 8) + (198R - 68)t + 71R - 32,\\
w_4(t) &= 2R(8 + (17 + 3R)t + 4R^2t^2).
\end{aligned}$$

Direct multiplication gives the exact polynomial identity

$$w(t)(tI - \mathcal{S}) = (Q_R(t), 0, 0, 0).$$

At $t = \rho$, this is a left ρ-eigenvector. The exterior eigenspace is one-dimensional, so the vector in Lemma 2 may be rescaled to $w(\rho)$, with the inverse rescaling absorbed into L_m. Since $R > 7$ and $\rho > 1$, every displayed grouping is positive. Thus $w_j(\rho) > 0$ for $j = 1, 2, 3, 4$; below we abbreviate $w_j = w_j(\rho)$.

Undoing the balancing in (36) gives, whenever $\Lambda(a) \neq 0$,

$$aG_m\mathbf{e}_j = (m!)^2 m^{-(j-1)}\Lambda(a)L_m\big(w_j + o_a(1)\big). \tag{41}$$

It remains to verify that the two official rows are not exceptional. The positivity estimate and $Q_m = x_0^{m+1} q_m$ give

$$\frac{q_m(x_0)}{(m!)^2} \geq 18(R-1)\,[29(R-1)]^m. \tag{42}$$

If $\Lambda(C) = 0$, the first coordinate of (35) would contradict (42). Therefore

$$\Lambda(A_1) = S\Lambda(C) \neq 0.$$

If $\Lambda(A_0) = 0$, then (35), (36), and $|d_m| > 1$ for large m would make the first-column quotient tend to zero, contradicting (33). Hence $\Lambda(A_0) \neq 0$ as well.

Because $w_j > 0$, equation (41) first proves that every $Q_{m,j}$ is nonzero for all sufficiently large m, and only then permits division:

$$\lim_{m\to\infty} \frac{P_{m,j}}{Q_{m,j}} = \frac{\Lambda(A_0)}{\Lambda(A_1)} = \lim_{m\to\infty} \frac{P_{m,1}}{Q_{m,1}} = \frac{\sqrt{10005}}{\pi}.$$

We conclude:

Theorem 1 (Ramanujan Challenge Problem 2.8)**.** *For every official column $j = 1, 2, 3, 4$,*

$$\boxed{\lim_{N\to\infty} \frac{P_{N,j}}{Q_{N,j}} = \frac{\sqrt{10005}}{\pi}.}$$

Equivalently,

$$\boxed{\lim_{N\to\infty} \frac{Q_{N,j}}{P_{N,j}} = \frac{\pi}{\sqrt{10005}}.}$$

10 Reproducibility map

The proof package contains the following certificates.

File	Exact obligation
`p28_full_closure_certificate.wl`	Optional independent Wolfram cross-check of the differential gauge and hypergeometric closure.
`p28_standalone_equations.py`	Mandatory dependency-free expansion of the four cleared Ore factorizations, the tail coefficient equations, the terminating base, generic, and top identities, ascension, and the ${}_3F_2$ equation.
`p28_dominant_product_algebra.py`	Mandatory dependency-free verification of $\mathcal{B}_m = \mathcal{S} + O(m^{-1})$, the characteristic polynomial, the root-separation inequalities, the left-eigenvector identity, and the four positive coordinate rewrites.
`p28_kernel_contiguity_certificate.sage`	Optional independent Sage check of the four displayed factorizations; it performs no Ore division.
`p28_lattice_hypotheses_certificate.sage`	Optional exact cross-check of the rank-one factorization, tail direction, and transformed ODE identities.
`p28_convergence_constants.py`	Exact rational verification of the coefficient bounds, $\alpha_{n+1}/\alpha_n \geq 29n^2$, and $\beta(x_0) < 1$.
`all_four_columns_certificate.sage`	Optional Sage cross-check of the balanced limit and exterior-root algebra; the analytic contraction is proved in Lemma 2.
`p28_parametric_pade_probe.py`	Diagnostic finite exact regression of the predicted valuations; it is not used as proof of an all-N statement.

The two mandatory algebra checkers use only the standard library's `fractions.Fraction`, sparse coefficient dictionaries, and explicit addition, multiplication, differentiation, and determinant expansion. They invoke no division algorithm, factorizer, root finder, special-function library, or numerical sample. Wolfram Language and SageMath are optional independent cross-checks, not a trust requirement.

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