An Exact Hypergeometric Tail Certificate for Ramanujan Challenge Problem 2.8

Problem 2.8 submission

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Abstract

Let $G_N = M_0 M_1 \cdots M_{N-1}$ be the 4×4 transfer product in Ramanujan Challenge Problem 2.8, and let $P_{N,j}$ and $Q_{N,j}$ be the two official seeded rows evaluated in column j. We prove

$$\lim_{N\to\infty} \frac{P_{N,j}}{Q_{N,j}} = \frac{\sqrt{10005}}{\pi} \qquad (j = 1, 2, 3, 4).$$

Equivalently, $Q_{N,j}/P_{N,j} \to \pi/\sqrt{10005}$.

The missing connection constant is fixed by an exact rank-three hypergeometric tail. A nonterminating ${}_4F_3$ Euler jet is carried backward by the displayed matrix, while the first denominator is a terminating adjoint ${}_4F_3$. Four denominator-cleared Ore identities and three coefficient identities give an all-N Padé divisibility theorem. Positivity at the negative CM point, a Cauchy estimate, and an elementary positive-cone contraction prove convergence for all four columns. Every algebraic identity used below is reproduced by a dependency-free rational-polynomial checker. The classical Chudnovsky formula, cited precisely in Section 2, is the sole imported theorem.

Contents

1 Statement and compact form of the seeds

Put

$$R = 151931373056001 = 53360^3 + 1, \qquad x_0 = \frac{1}{R}, \qquad z = -\frac{x}{1-x}.$$

Thus the official CM point is

$$z_0 = -\frac{1}{R-1} = -\frac{1}{53360^3}.$$

Let

$$G_N = M_0 M_1 \cdots M_{N-1}, \qquad G_0 = I_4,$$

where the rational family is defined as follows. Put $r = x^{-1}$, $\omega = u(3u-2)(3u+2)$, and

$$\begin{aligned}
a_1 &= r(144u^5 - 288u^4 + 144u^3) - 99u^5 + 333u^4 - 229u^3 - 114u^2 + 40u + 64,\\
a_2 &= r(432u^4 - 864u^3 + 432u^2) - 243u^4 + 909u^3 - 868u^2 - 80u + 272,\\
a_3 &= r(432u^3 - 864u^2 + 432u) - 153u^3 + 648u^2 - 860u + 360,\\
a_4 &= 144r(u-1)^2,\\
b_1 &= -144ru^3 + 9u^4 + 63u^3 + 158u^2 + 168u + 64,\\
b_2 &= 216ru^2 + 36u^3 - 189u^2 - 316u - 168,\\
b_3 &= 108ru + 54u^2 - 189u - 158,\\
c_1 &= -288r^2u^3 + r(54u^4 + 378u^3 + 948u^2 + 1008u + 384)\\
&\qquad + 18u^5 + 45u^4 - 251u^3 - 1086u^2 - 1384u - 576,\\
c_2 &= -432r^2u^2 + r(153u^4 - 657u^3 + 1292u^2 + 2064u + 1072)\\
&\qquad - 72u^4 + 702u^3 - 1069u^2 - 2508u - 1512,\\
c_3 &= -216r^2u + r(180u^3 - 891u^2 + 1450u + 1116)\\
&\qquad - 108u^3 + 864u^2 - 1385u - 1422,\\
c_4 &= -4r^2 + r(6u^2 - 33u + \tfrac{536}{9}) - 4u^2 + 32u - 63.
\end{aligned}$$

Define

$$\mathcal{M}(u,x) = \begin{pmatrix} a_1/\omega & a_2/\omega & a_3/\omega & a_4/\omega \\ -u^3 & -3u^2 & -3u & -1 \\ xb_1/144 & -xb_2/72 & -xb_3/36 & x(-2r-(2u-7))/2 \\ x^2c_1/288 & x^2c_2/144 & x^2c_3/72 & x^2c_4/4 \end{pmatrix}, \qquad M_N(x) = \mathcal{M}(2N+3, x). \tag{1}$$

At $x = x_0$, the only deformed challenge coefficient is restored by

$$236337691420383 = \frac{14R - 567}{9}.$$

Thus every use of Cauchy's theorem below concerns the explicitly displayed rational x-family, not an unspecified continuation.

Define four Pascal rows

$$\begin{aligned} b_0 &= (1,0,0,0), \quad b_1 = (1,1,0,0),\\ b_2 &= (1,2,1,0), \quad b_3 = (1,3,3,1) \end{aligned}$$

and let $\mathcal{P}$ be the matrix with rows b_0, b_1, b_2, b_3. The compact denominator row is

$$C(x) = \left(\frac{18}{x} + \frac{159}{4},\ \frac{54}{x} + \frac{131}{2},\ \frac{54}{x} + 27,\ \frac{18}{x}\right).$$

Equivalently,

$$C = \frac{18}{x} b_3 + \frac{1}{4}(5b_0 + 46b_1 + 108b_2).$$

Set

$$A = 13591409, \qquad B = 545140134, \qquad S = 426880.$$

The two official initial rows have the exact form

$$A_1 = SC, \qquad A_0 = AC - \frac{5}{4} H_0, \qquad H_0 = Ab_0 + Bb_1 = (A + B, B, 0, 0). \tag{2}$$

At $x = x_0$, these identities reproduce the official integer rows entry by entry:

$$\begin{aligned} A_0 = (&37169305760442252761441,\ 111507917281327441564208,\\ &111507917281327599720129,\ 37169305760442410917362),\\ A_1 = (&11674163615426396923\,20,\ 35022490846278961321\,60,\\ &35022490846278796972\,80,\ 11674163615426227238\,40).\end{aligned}$$

For $j = 1, \ldots, 4$, write

$$P_{N,j} = A_0 G_N \mathbf{e}_j, \qquad Q_{N,j} = A_1 G_N \mathbf{e}_j.$$

We first identify the first-column limit and then use an exact positive-cone transfer to cover all four columns.

2 The CM function and its exact value

Let

$$y(z) = {}_3F_2\left(\begin{matrix}\frac{1}{6}, \frac{1}{2}, \frac{5}{6} \\ 1, 1\end{matrix}; z\right), \qquad \theta = z\frac{d}{dz} = (1 - x)x\frac{d}{dx},$$

and define

$$\Phi(x) = \frac{Ay(z) + B\theta y(z)}{S}. \tag{3}$$

We use one external theorem: the classical Chudnovsky identity. In the normalization used here it is Theorem 0.1 of Milla's equation-by-equation derivation [2], whose modular and CM proof is completed in Theorem 9.7 and Chapter 10:

$$\frac{1}{\pi} = \frac{12}{640320^{3/2}} \sum_{k=0}^{\infty} \frac{(6k)!}{(3k)!(k!)^3}(A + Bk)(-640320^{-3})^k.$$

The elementary coefficient identity

$$\frac{(6k)!}{(3k)!(k!)^3} = 1728^k \frac{(\frac{1}{6})_k(\frac{1}{2})_k(\frac{5}{6})_k}{(k!)^3}$$

and $640320 = 12 \cdot 53360$ give

$$Ay(z_0) + B\theta y(z_0) = \frac{640320^{3/2}}{12\pi} = \frac{426880\sqrt{10005}}{\pi}.$$

Consequently,

$$\boxed{\Phi(x_0) = \frac{\sqrt{10005}}{\pi}.} \tag{4}$$

3 The nonterminating adjoint tail

Put $n = N+1$, $\delta_N = \theta - n$, and

$$\nu_n = \frac{576n^2(2n+1)^2}{(6n+1)(6n+5)}. \tag{5}$$

Define

$$F_N(z) = \kappa_N z^n {}_4F_3\left(\begin{matrix} n, n+\frac{1}{6}, n+\frac{1}{2}, n+\frac{5}{6} \\ 2n, 2n, 2n \end{matrix}; z\right), \tag{6}$$

where

$$\kappa_0 = \frac{5}{72}, \qquad \frac{\kappa_{N+1}}{\kappa_N} = -\nu_{N+1}^{-1}.$$

Its Euler jet is

$$k_N = \left(F_N, \delta_N F_N, \delta_N^2 F_N, \delta_N^3 F_N\right)^T.$$

Proposition 1 (Exact tail contiguity)**.** *For every $N \geq 0$,*

$$\boxed{M_N k_{N+1} = k_N.} \tag{7}$$

Proof. Put $m = (u-1)/2 = N+1$. For the r-th row of $\mathcal{M}(u,x)$, define

$$P_r(t) = \sum_{s=0}^{3} \mathcal{M}(u,x)_{r+1,s+1} t^s, \qquad \mathcal{T}Q = (1-x)x\partial_x Q + (t+1)Q.$$

The shifted tail satisfies

$$L_+(t) F_{N+1} = 0,$$

where

$$L_+(t) = (1-x)t(t+u)^3 + x(t+m+1)(t+m+\tfrac{7}{6})(t+m+\tfrac{3}{2})(t+m+\tfrac{11}{6}).$$

There is no division step: direct expansion gives the following four denominator-cleared polynomial identities:

$$u(3u-2)(3u+2)x(\mathcal{T}P_0 - P_1) = 144(u-1)^2 L_+, \tag{8}$$

$$\mathcal{T}P_1 - P_2 = -L_+, \tag{9}$$

$$2(\mathcal{T}P_2 - P_3) = (-2+7x-2ux)L_+, \tag{10}$$

$$36(\mathcal{T}P_3 + \ell_3 P_3 + \ell_2 P_2 + \ell_1 P_1 + \ell_0 P_0) = q_3 L_+, \tag{11}$$

with

$$\begin{aligned}
\ell_0 &= (u-1)u(3u-2)(3u+2)x/144, \\
\ell_1 &= [-576 + 864u - 432u^2 + 72u^3 + (580 - 872u + 405u^2 - 36u^3)x]/72, \\
\ell_2 &= [432 - 432u + 108u^2 + (-436 + 405u - 54u^2)x]/36, \\
\ell_3 &= (-12 + 6u + 11x - 2ux)/2, \\
q_3 &= -36 + (536 - 297u + 54u^2)x + (-567 + 288u - 36u^2)x^2.
\end{aligned}$$

For clarity, the coefficient calculation producing the first row is also written without a special-function routine. Set

$$A(t) = \frac{144(u-1)^2(t+u)^3}{u(3u-2)(3u+2)}, \qquad B(t) = P_0(t) - \frac{A(t)}{x},$$

$$\varrho = -\nu_m^{-1} = -\frac{(3u-2)(3u+2)}{144(u-1)^2u^2},$$

$$\chi_j = \frac{(m+j)(m+\frac{1}{6}+j)(m+\frac{1}{2}+j)(m+\frac{5}{6}+j)}{m(m+\frac{1}{6})(m+\frac{1}{2})(m+\frac{5}{6})} \times \left(\frac{2m(2m+1)}{(2m+j)(2m+j+1)}\right)^3,$$

$$\psi_j = \frac{(m+j)(m+\frac{1}{6}+j)(m+\frac{1}{2}+j)(m+\frac{5}{6}+j)}{(2m+j)^3(j+1)}.$$

The constant and generic coefficient equations are

$$\varrho[-A(0)] = 1, \qquad \varrho\left[-A(j)\chi_j + \frac{(A(j-1)+B(j-1))\chi_{j-1}}{\psi_{j-1}}\right] = 1 \quad (j \geq 1). \tag{12}$$

Clearing the displayed nonzero factors turns (8)–(12) into polynomial equalities with every coefficient zero. The independent script `p28_standalone_equations.py` expands precisely these equalities using only rational addition and multiplication. Equation (12) gives the first row of (7); applying (8)–(11) successively gives the other three Euler-jet rows. □

For $N = 0$, write $y(z) = 1 + \sum_{k\geq 1} c_k z^k$. The coefficient of z is

$$c_1 = \frac{(1/6)(1/2)(5/6)}{1^3} = \frac{5}{72},$$

and, after shifting $k \mapsto k+1$, both sides below have initial coefficient $5/72$ and consecutive-coefficient ratio

$$\frac{(k+\frac{1}{6})(k+\frac{1}{2})(k+\frac{5}{6})}{(k+1)^3}.$$

Hence coefficient equality, rather than a named ascension rule, gives

$$F_0 = y - 1 = \frac{5}{72} z\, {}_4F_3\left(\begin{matrix}1, \frac{7}{6}, \frac{3}{2}, \frac{11}{6} \\ 2, 2, 2\end{matrix}; z\right). \tag{13}$$

Because $\mathcal{P}$ is the Pascal matrix,

$$\mathcal{P}k_0 = (y-1, \theta y, \theta^2 y, \theta^3 y)^T.$$

4 The rank-three error carrier

Set

$$f = (y-1, \theta y, \theta^2 y, \theta^3 y)^T, \qquad \mathcal{E}_0 = \frac{5}{4}\mathcal{P} + fC, \qquad \mathcal{E}_N = \mathcal{E}_0 G_N.$$

The transformed hypergeometric equation is

$$72\theta^3 y + 108x\theta^2 y + 46x\theta y + 5xy = 0. \tag{14}$$

Indeed the coefficient ratio of y gives

$$\left[\theta^3 - z(\theta + \tfrac{1}{6})(\theta + \tfrac{1}{2})(\theta + \tfrac{5}{6})\right] y = 0.$$

Substituting $z = -x/(1-x)$, using $\theta = (1-x)x\partial_x$, and multiplying by $72(1-x)$ expands to (14). Using the displayed decomposition of C, equations (13)–(14) give

$$Ck_0 = -\frac{5}{4}.$$

It follows that

$$\mathcal{E}_0 k_0 = \frac{5}{4} f + f(Ck_0) = 0.$$

Proposition 1 therefore implies

$$\boxed{\mathcal{E}_N k_N = 0 \qquad (N \geq 0).} \tag{15}$$

The same differential equation gives the exact row relation

$$72(\mathcal{E}_N)_{3,*} + 108x(\mathcal{E}_N)_{2,*} + 46x(\mathcal{E}_N)_{1,*} + 5x(\mathcal{E}_N)_{0,*} = 0. \tag{16}$$

5 A discrete valuation lemma

Let

$$H = \operatorname{diag}(x, 1, 1, 1), \qquad J_N = HM_N.$$

Every entry of J_N is regular at $x = 0$. If $u = 2N + 3$ and

$$a_N = \frac{144(u-1)^2}{u(3u-2)(3u+2)}, \qquad V_N = (u^3, 3u^2, 3u, 1),$$

direct substitution in the authoritative matrix gives

$$J_N(0) = \begin{pmatrix} a_N \\ -1 \\ -1 \\ -1 \end{pmatrix} V_N. \tag{17}$$

Thus $J_N(0)$ has rank one.

The first nonconstant coefficient of the ${}_4F_3$ in (6) equals

$$c_N = \frac{u(3u-2)(3u+2)}{144(u-1)^2} = a_N^{-1}.$$

Since $z = -x + O(x^2)$,

$$Hk_N = x^{N+2}\eta_N \left[\begin{pmatrix} 1 \\ -c_N \\ -c_N \\ -c_N \end{pmatrix} + O(x) \right], \qquad \eta_N \neq 0. \tag{18}$$

The leading vector in (18) is precisely the image direction in (17). The other expansion required below is equally direct. Since

$$F_{N+1} = \kappa_{N+1} z^{N+2}(1 + O(z)), \qquad \delta_{N+1} = \theta - (N+2),$$

the shifted derivatives kill the leading monomial, and therefore

$$k_{N+1} = x^{N+2}\widetilde{\eta}_N \left(\mathbf{e}_1 + xs_N + O(x^2)\right), \qquad \widetilde{\eta}_N \neq 0. \tag{19}$$

Moreover, (17) gives

$$J_N(0)\mathbf{e}_1 = u^3(a_N, -1, -1, -1)^T \neq 0.$$

Lemma 1 (DVR step, including the extra first-column zero)**.** *Let* $R_0 = \mathbb{Q}[[x]]$*,* $H = \mathrm{diag}(x, 1, 1, 1)$*, and suppose*

$$E = x^N LH, \qquad L \in \mathrm{Mat}_4(R_0), \qquad Ek = 0.$$

Assume $J = HM \in \mathrm{Mat}_4(R_0)$*,* $Mk^+ = k$*, and*

$$k^+ = x^r\alpha(\mathbf{e}_1 + xs + O(x^2)),$$
$$Hk = x^r\beta(v_0 + xv_1 + O(x^2)),$$

with $\alpha\beta \neq 0$*. If* $J(0)$ *has rank one,* $\mathrm{im}\, J(0) = \mathbb{Q}v_0$*, and* $J(0)\mathbf{e}_1 \neq 0$*, then*

$$EM = x^{N+1}L^+H$$

for some $L^+ \in \mathrm{Mat}_4(R_0)$*.*

Proof. Absorb β/α into v_0, v_1. From $Jk^+ = Hk$,

$$J(\mathbf{e}_1 + xs + O(x^2)) = v_0 + xv_1 + O(x^2).$$

Write $L = L_0 + xL_1 + \cdots$. The equation $L(Hk) = 0$ gives

$$L_0v_0 = 0, \qquad L_0v_1 + L_1v_0 = 0.$$

Since $J(0)$ has image $\mathbb{Q}v_0$, $L_0J(0) = 0$, so LJ is entrywise divisible by x. The coefficient of x in its first column is

$$L_0(v_1 - J(0)s) + L_1v_0 = -L_0J(0)s = 0.$$

Hence that column is divisible by x^2. Therefore

$$L^+ = x^{-1}LJH^{-1}$$

is regular and $EM = x^{N+1}L^+H$. □

Proposition 2 (All-N Padé divisibility)**.** *For every* $N \geq 0$*, there is* $L_N \in \mathrm{Mat}_4(\mathbb{Q}[[x]])$ *such that*

$$\boxed{\mathcal{E}_N = x^N L_N H.} \tag{20}$$

Thus every row of $\mathcal{E}_N$ *has componentwise valuations at least*

$$(N+1, N, N, N).$$

The last row has the stronger valuations

$$(N+2, N+1, N+1, N+1).$$

Proof. Every component of f is $O(x)$, while C has only a simple pole. All four components of f have leading term $-5x/72$. Consequently the constant term in the first column of fC is $-5/4$, cancelling the first component of every row of $(5/4)\mathcal{P}$. Hence $\mathcal{E}_0 = L_0H$.

Apply Lemma 1 inductively, using (15), (17), (18), (19), the displayed nonzero first column of $J_N(0)$, and $M_Nk_{N+1} = k_N$. The stronger last-row assertion follows from (16). □

6 The terminating denominator

For the first column put

$$q_N(x) = CG_N\mathbf{e}_1, \qquad n = N+1, \qquad Q_N(x) = x^n q_N(x),$$

and define

$$\widehat{Q}_N(z) = (1-z)^n Q_N\left(-\frac{z}{1-z}\right).$$

Since $x = -z/(1-z)$, this is also the first component of $(-z)^n CG_N$.

Proposition 3 (Exact terminating denominator)**.** *For every $N \geq 0$,*

$$\frac{\widehat{Q}_N(z)}{\alpha_n} = {}_4F_3\left(\begin{matrix}-n, -n-\frac{1}{6}, -n-\frac{1}{2}, -n-\frac{5}{6} \\ 1-2n, 1-2n, 1-2n\end{matrix}; z\right), \tag{21}$$

where

$$\alpha_1 = 18, \qquad \frac{\alpha_{n+1}}{\alpha_n} = \nu_n. \tag{22}$$

Here and below the hypergeometric expression denotes the unambiguous finite sum over $0 \leq k \leq n$; it terminates before any lower Pochhammer symbol can vanish.

Proof. Write $p_n = \widehat{Q}_{n-1}$ for $n \geq 1$. We use coefficient induction, because normalization at $z = 0$ alone would not select a unique solution of the fourth-order equation. To expose the matrix-to-scalar bridge, write

$$\mathcal{L}_n(t) = t(t+2n-1)^3 - z(t+n)(t+n+\tfrac{1}{6})(t+n+\tfrac{1}{2})(t+n+\tfrac{5}{6}) = \sum_{j=0}^{4} c_j t^j$$

and define, for operator polynomials with coefficients on the left,

$$\Theta Q = z\partial_z Q + tQ.$$

The four horizontal components are reconstructed from the first by

$$\pi_0 = 1, \qquad \pi_3 = \frac{c_4}{c_0}t, \qquad \pi_2 = \frac{c_3}{c_4}\pi_3 - \Theta\pi_3, \qquad \pi_1 = \frac{c_2}{c_4}\pi_3 - \Theta\pi_2.$$

The remaining horizontal equation is the direct factorization

$$\begin{aligned}\Theta\pi_1 + 1 - \frac{c_1}{c_4}\pi_3 = {} & -\frac{72}{n(2n+1)(6n+1)(6n+5)z}\big[t(t-2n)^3 \\ & \qquad - z(t-n)(t-n-\tfrac{1}{6})(t-n-\tfrac{1}{2})(t-n-\tfrac{5}{6})\big].\end{aligned}$$

Let

$$\mathcal{C}_n(z) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -c_0/c_4 & -c_1/c_4 & -c_2/c_4 & -c_3/c_4 \end{pmatrix}.$$

The propagation of the reconstructed row is the following sixteen-entry identity:

$$\mathcal{C}_n[-z\mathcal{M}(2n+1, -z/(1-z))] - \theta[-z\mathcal{M}(2n+1, -z/(1-z))] - [-z\mathcal{M}(2n+1, -z/(1-z))]\mathcal{C}_{n+1} = 0. \tag{23}$$

Finally, direct contraction with the displayed matrix gives

$$\sum_{r=0}^{3} \pi_r(t)\,[-z\mathcal{M}(2n+1,-z/(1-z))]_{r+1,1} = d_0(t) + zd_1(t). \tag{24}$$

These are identities in $\mathbb{Q}(n,z)(t)$. The factors depending on the integer n are nonzero for $n \geq 1$; no value is obtained by dividing at $z=0$, because the verifier cross-multiplies first and the reconstructed base functions have removable limits there. The mandatory sparse-polynomial verifier checks every reconstruction equation, the closing factorization, and (23)–(24) by cross multiplication. For any reconstructed horizontal row these identities give

$$p_{n+1} = \big(d_0(\theta) + zd_1(\theta)\big)p_n, \tag{25}$$

where

$$u = 2n+1, \qquad q = u-1-t = 2n-t,$$
$$d_0(t) = \frac{144u^2q^3}{(u-1)(3u-2)(3u+2)}$$

and

$$d_1(t) = -\frac{\widehat{P}(u,q)}{(u-1)u(3u-2)(3u+2)},$$

with

$$\begin{aligned}\widehat{P}(u,q) = 144u^3q^3uq(&63u^4 - 315u^3 + 474u^2 - 168u - 64)\\ &+ q^2(-153u^4 + 441u^3 - 158u^2 - 168u - 64)\\ &- (u-2)(u-1)u^2(3u-8)(3u-4).\end{aligned}$$

The standalone checker expands this compact expression back to twice the former $P(n,t)$, coefficient by coefficient. Thus (25) is the scalar form of the displayed matrix recurrence; no differential-equation uniqueness is used below.

Let

$$h_{n,k} = \frac{(-n)_k(-n-\frac{1}{6})_k(-n-\frac{1}{2})_k(-n-\frac{5}{6})_k}{(1-2n)_k^3 k!} \quad (0 \leq k \leq n), \qquad \nu_n \text{ is given by (5).}$$

For $n \geq 1$, clearing the displayed nonzero denominators gives exactly

$$d_0(0) = \nu_n, \tag{26}$$
$$d_0(k) + d_1(k-1)\frac{h_{n,k-1}}{h_{n,k}} = \nu_n \frac{h_{n+1,k}}{h_{n,k}} \qquad (1 \leq k \leq n), \tag{27}$$
$$d_1(n) = -\nu_n \frac{(n+\frac{7}{6})(n+\frac{3}{2})(n+\frac{11}{6})}{8(2n+1)^3}. \tag{28}$$

The base row is not assumed: from the first component of C,

$$q_0(x) = \frac{18}{x} + \frac{159}{4}, \qquad Q_0(x) = xq_0(x) = 18 + \frac{159}{4}x,$$

and therefore

$$p_1(z) = (1-z)Q_0\Big(-\frac{z}{1-z}\Big) = 18\left(1 - \frac{77}{24}z\right).$$

Direct application of the four displayed π_r operators gives the four base identities

$$-zC\left(-\frac{z}{1-z}\right) = \left.(\pi_0(\theta)p_1, \pi_1(\theta)p_1, \pi_2(\theta)p_1, \pi_3(\theta)p_1)\right|_{n=1}. \tag{29}$$

The fourth base equation is also direct:

$$\left[\theta(\theta-2)^3 - z(\theta-1)(\theta-\tfrac{7}{6})(\theta-\tfrac{3}{2})(\theta-\tfrac{11}{6})\right] p_1 = 0.$$

Equivalently, if $H_1 = -zC(-z/(1-z))$, then all four components of

$$\theta H_1 + H_1 \mathcal{C}_1 = 0$$

vanish. The mandatory verifier checks both forms independently. Moreover,

$$\deg((d_0(\theta) + zd_1(\theta))p_n) \le n+1.$$

Equations (26)–(28) therefore prove, coefficient by coefficient, that the right side of (25) is $\alpha_{n+1} \sum_{k=0}^{n+1} h_{n+1,k} z^k$. Starting from (29), (23) propagates the horizontal form at every step. This proves (21) and (22) for the actual row CG_N, not merely for a scalar surrogate. The mandatory standalone checker independently expands the cleared identities and rejects any nonzero coefficient. □

The two normalizations are reciprocal at every step:

$$\boxed{\kappa_N \alpha_{N+1} = (-1)^N \frac{5}{4}.} \tag{30}$$

Indeed it holds at $N=0$, and the ratios $\kappa_{N+1}/\kappa_N = -\nu_{N+1}^{-1}$ and $\alpha_{N+2}/\alpha_{N+1} = \nu_{N+1}$ change only its sign.

Corollary 1 (Positivity at the CM point)**.** *At $z_0 = -1/53360^3$,*

$$\widehat{Q}_N(z_0) \ge \alpha_n > 0.$$

Moreover,

$$\alpha_n \ge 18 \cdot 64^N (N!)^2.$$

Proof. For $0 \le k \le n$, the coefficient of z^k in (21) has sign $(-1)^k$. Since $z_0 < 0$, every summand is nonnegative. Also

$$\nu_n - 64n^2 = \frac{256n^2}{(6n+1)(6n+5)} > 0.$$

Iterating (22) proves the lower bound. □

7 From formal contact to convergence at x_0

This step is included to rule out a beyond-all-orders ambiguity. For $r = 0, 1$, let

$$E_{N,r}(x) = (\mathcal{E}_N)_{r,1}, \qquad \mathcal{R}_{N,r}(x) = x^n E_{N,r}(x).$$

Proposition 2 says that $\mathcal{R}_{N,r}$ has a zero of order at least $2n$.

Choose $r_0 = 1/4$. On $|x| = r_0$, $|z| \le 1/3$. The coefficients of y have modulus at most one, so

$$(|y-1|, |\theta y|, |\theta^2 y|, |\theta^3 y|) \le \left(\frac{1}{2}, \frac{3}{4}, \frac{3}{2}, \frac{33}{8}\right).$$

Using the four row sums $1, 2, 4, 8$ of $\mathcal{P}$ and the displayed compact row C, this gives

$$\|\mathcal{E}_0\|_\infty \le \frac{93809}{32} < 3000.$$

For $m \ge 1$, put

$$D(m) = \operatorname{diag}(1, m, m^2, m^3), \qquad \mathcal{B}_m = D(m)^{-1} M_m D(m+1)/(m+1)^2.$$

On $|x| = 1/4$, direct sums of the absolute coefficients in the authoritative entries give

$$\|M_0\|_\infty \le 8203 < 9000.$$

For the balanced transfer, $u/m \le 5$ and $u/(m+1) \le 5/2$. In its fourth column the negative power is combined with the row power before bounding:

$$\frac{m+1}{u} < \frac{1}{2}, \qquad \frac{m+1}{m} \le 2, \qquad \frac{u(m+1)}{m^2} \le 10, \qquad \frac{u^2(m+1)}{m^3} \le 50.$$

The last two inequalities follow after clearing $m > 0$ from

$$10m^2 - u(m+1) = (m-1)(8m+3), \qquad 50m^3 - u^2(m+1) = (m-1)(46m^2 + 30m + 9).$$

The resulting four exact row bounds are

$$\frac{209299}{32}, \quad \frac{343}{4}, \quad \frac{694145}{1152}, \quad \frac{4981375}{512}.$$

Consequently

$$\|\mathcal{B}_m\|_\infty \le \frac{4981375}{512} < 10000.$$

The balancing telescopes:

$$G_N = M_0 (N!)^2 \mathcal{B}_1 \cdots \mathcal{B}_{N-1} D(N)^{-1}.$$

It follows that, for $N \ge 1$,

$$\max_{|x|=1/4} |\mathcal{R}_{N,r}(x)| < 27{,}000{,}000\,(N!)^2\, 10000^{N-1} (1/4)^n. \tag{31}$$

We now verify the analytic hypothesis behind the next step. On $|x| \le 1/4$,

$$\left|-\frac{x}{1-x}\right| \le \frac{1}{3} < 1,$$

so the series defining y and its first three Euler derivatives are holomorphic on a neighborhood of the closed disk. Because $f = O(x)$, the simple pole of C cancels in fC, so $\mathcal{E}_0$ is holomorphic there. Inspection of (1) shows that each M_m is holomorphic off $x = 0$ in this disk and has at most a simple pole at 0. Hence $\mathcal{R}_{N,r} = x^n (\mathcal{E}_N)_{r,1}$, with $n = N + 1$, is holomorphic on the disk after removing its possible singularity at zero. Proposition 2 strengthens its order there to at least $2n$. Therefore

$\mathcal{R}_{N,r}(x)/x^{2n}$ has a removable singularity at zero and is holomorphic on a neighborhood of the closed disk. Applying the maximum principle to $\mathcal{R}_{N,r}(x)/x^{2n}$ gives

$$|\mathcal{R}_{N,r}(x_0)| < 27{,}000{,}000\,(N!)^2\,10000^{N-1}(1/4)^n(4x_0)^{2n}. \tag{32}$$

Because $1-z = 1/(1-x)$,

$$Q_N(x_0) = (1-x_0)^n \widehat{Q}_N(z_0).$$

Corollary 1 yields

$$Q_N(x_0) \geq 18 \cdot 64^N (N!)^2 (1-x_0)^n.$$

Combining this with (32), we obtain

$$\left|\frac{E_{N,r}(x_0)}{q_N(x_0)}\right| = \left|\frac{\mathcal{R}_{N,r}(x_0)}{Q_N(x_0)}\right| \leq K_0\,\beta(x_0)^N, \tag{33}$$

where $K_0 < \infty$ is independent of N and

$$K_0 = \frac{600}{R(R-1)}, \qquad \beta(x_0) = \frac{10000}{64}\,\frac{x_0^2}{(1/4)(1-x_0)} = \frac{625}{R(R-1)} < 3 \cdot 10^{-26} < 1.$$

Therefore

$$\frac{E_{N,0}(x_0)}{q_N(x_0)} \longrightarrow 0, \qquad \frac{E_{N,1}(x_0)}{q_N(x_0)} \longrightarrow 0. \tag{34}$$

8 Identification of the first-column limit

By (2) and the definition of Φ,

$$A_0 - \Phi A_1 = -A(\mathcal{E}_0)_{0,*} - B(\mathcal{E}_0)_{1,*}.$$

Multiplying by $G_N\mathbf{e}_1$, dividing by $A_1G_N\mathbf{e}_1 = Sq_N$, and using (34), we get

$$\lim_{N\to\infty} \frac{A_0G_N\mathbf{e}_1}{A_1G_N\mathbf{e}_1} = \Phi(x_0).$$

Equation (4) therefore proves

$$\boxed{\lim_{N\to\infty} \frac{P_{N,1}}{Q_{N,1}} = \frac{\sqrt{10005}}{\pi}.} \tag{35}$$

9 The other three official columns

All matrices in this section are evaluated at $x = x_0 = 1/R$. For $m \geq 1$, retain

$$D_m = \mathrm{diag}(1, m, m^2, m^3), \qquad \mathcal{B}_m = D_m^{-1} M_m D_{m+1}/(m+1)^2$$

and put

$$Z_m(a) = \frac{aG_mD_m}{(m!)^2}, \qquad Y_m(a) = Z_m(a)\mathcal{P}^{-1}, \qquad T_m = \mathcal{P}\mathcal{B}_m\mathcal{P}^{-1}.$$

The balancing gives the exact row recurrence

$$Y_{m+1}(a) = Y_m(a)T_m. \tag{36}$$

Set $k = m - 1$, $s = R - 4$, and

$$g_m = (2m + 3)(6m + 7)(6m + 11).$$

Direct cross multiplication of the authoritative matrix gives

$$(T_m)_{ij} = \frac{N_{ij}(k, s)}{D_{ij}(m, R)}, \tag{37}$$

where

$$(D_{ij}) = \begin{pmatrix} (m+1)^2 g_m & (m+1) g_m & g_m & g_m \\ m g_m & m(m+1) g_m & m g_m & m g_m \\ 24m^2(m+1)^2 g_m R & 72m^2(m+1) g_m R & 36m^2 g_m R & 2m^2 g_m R \\ 48m^3(m+1)^2 g_m R^2 & 144m^3(m+1) g_m R^2 & 72m^3 g_m R^2 & 36m^3 g_m R^2 \end{pmatrix}.$$

Each numerator has the form

$$N_{ij}(k, s) = \sum_{a,b} c_{ab}^{(ij)} k^a s^b, \qquad c_{ab}^{(ij)} > 0.$$

For example,

$$\begin{aligned} N_{11} = {} & 209067 + 62208s + (409482 + 124416s)k \\ & + (318165 + 98496s)k^2 + (122806 + 38592s)k^3 \\ & + (23580 + 7488s)k^4 + (1800 + 576s)k^5. \end{aligned}$$

The complete finite list of all 285 positive integers $c_{ab}^{(ij)}$ is printed in `POSITIVE_CONE_CERTIFICATE.md`. The dependency-free verifier `p28_positive_cone.py` reconstructs $M_m, \mathcal{B}_m, T_m$, cross-multiplies every one of the sixteen identities (37), and compares every coefficient with that list. Thus, without sampling or a positivity oracle,

$$\boxed{T_m > 0 \quad \text{entrywise for every } m \geq 1,\ R \geq 4.} \tag{38}$$

Leading coefficients in k, checked by the same exact arithmetic, give

$$\widetilde{\mathcal{S}} := \lim_{m\to\infty} T_m = \begin{pmatrix} 8R-7 & 4(6R-5) & 24R-17 & 8R \\ 8(R-1) & 24R-23 & 4(6R-5) & 8R-1 \\ \frac{(R-1)(8R-1)}{R} & \frac{2(R-1)(12R-1)}{R} & 24R-23 & \frac{2(4R^2-R-1)}{R} \\ \frac{2(R-1)(4R^2-R-1)}{R^2} & \frac{(R-1)(24R^2-5R-4)}{R^2} & \frac{2(R-1)(12R-1)}{R} & \frac{8R^3-3R^2-4}{R^2} \end{pmatrix}.$$

Every displayed entry is positive for $R \geq 4$.

Both official rows enter this cone after the first transfer. Indeed $D_1 = I$, $G_1 = M_0$, and exact expansion gives

$$\begin{aligned} Y_1(A_1) = \Big(& \frac{320160}{77}(451657 + 259168s + 36864s^2), \\ & \frac{213440}{77}(1045771 + 591288s + 82944s^2), \\ & \frac{3841920}{77}(30075 + 16706s + 2304s^2), \\ & \frac{7683840}{77}(2612 + 1421s + 192s^2)\Big) \end{aligned}$$

and

$$Y_1(A_0) = \Big(\frac{13563858344917 + 18828949838688s + 4509303312384s^2}{924},$$
$$\frac{2(2606908232573 + 3613607517834s + 845494371072s^2)}{231},$$
$$\frac{3584820267815 + 4955797147464s + 1127325828096s^2}{308},$$
$$\frac{3(103400761441 + 142363659388s + 31314606336s^2)}{154}\Big).$$

Hence

$$Y_m(A_0) > 0, \qquad Y_m(A_1) > 0 \qquad (m \geq 1). \tag{39}$$

Lemma 2 (Elementary positive-cone contraction)**.** *For every $j = 1, 2, 3, 4$, the quotient*

$$\frac{A_0 G_m \mathbf{e}_j}{A_1 G_m \mathbf{e}_j}$$

is defined for $m \geq 1$, and all four quotients have one common limit.

Proof. Write

$$p_m = Y_m(A_0), \qquad q_m = Y_m(A_1), \qquad r_{m,i} = \frac{p_{m,i}}{q_{m,i}}.$$

Equations (36) and (39) give

$$r_{m+1,j} = \sum_{i=1}^{4} \omega_{ij}^{(m)} r_{m,i}, \qquad \omega_{ij}^{(m)} = \frac{q_{m,i}(T_m)_{ij}}{\sum_{h=1}^{4} q_{m,h}(T_m)_{hj}},$$

where

$$\omega_{ij}^{(m)} > 0, \qquad \sum_{i=1}^{4} \omega_{ij}^{(m)} = 1.$$

Since $T_m \to \widetilde{\mathcal{S}} > 0$, there are m_0 and $0 < a < b$ such that

$$a \leq (T_m)_{ij} \leq b \qquad (m \geq m_0;\ 1 \leq i, j \leq 4).$$

One such positive step implies

$$\frac{a}{b} \leq \frac{q_{m+1,i}}{q_{m+1,j}} \leq \frac{b}{a}.$$

Consequently, for $m \geq m_0 + 1$,

$$\omega_{ij}^{(m)} \geq \delta := \frac{a^2}{4b^2} > 0.$$

Let

$$\ell_m = \min_i r_{m,i}, \qquad u_m = \max_i r_{m,i}.$$

Every $r_{m+1,j}$ is a convex combination of the preceding four ratios, so ℓ_m is nondecreasing and u_m is nonincreasing. The weights on indices attaining the two endpoints are at least δ, whence

$$\ell_m + \delta(u_m - \ell_m) \leq r_{m+1,j} \leq u_m - \delta(u_m - \ell_m)$$

and

$$u_{m+1} - \ell_{m+1} \leq (1 - 2\delta)(u_m - \ell_m).$$

Thus all four $r_{m,i}$ tend to one positive constant c.

Finally $Z_m(a) = Y_m(a)\mathcal{P}$. Every column of $\mathcal{P}$ is nonzero and nonnegative, so

$$\frac{Z_m(A_0)_j}{Z_m(A_1)_j} = \frac{\sum_i q_{m,i}(\mathcal{P})_{ij} r_{m,i}}{\sum_i q_{m,i}(\mathcal{P})_{ij}}$$

is defined and is another convex combination of the $r_{m,i}$. It tends to c. Since

$$Z_m(a)_j = \frac{m^{j-1}}{(m!)^2} a G_m \mathbf{e}_j,$$

the same is true of the four official quotients. $\square$

The first-column identity (35) fixes their common value:

$$\lim_{m\to\infty} \frac{P_{m,j}}{Q_{m,j}} = \frac{\sqrt{10005}}{\pi} \qquad (j = 1, 2, 3, 4).$$

Theorem 1 (Ramanujan Challenge Problem 2.8)**.** *For every official column $j = 1, 2, 3, 4$,*

$$\boxed{\lim_{N\to\infty} \frac{P_{N,j}}{Q_{N,j}} = \frac{\sqrt{10005}}{\pi}.}$$

Equivalently,

$$\boxed{\lim_{N\to\infty} \frac{Q_{N,j}}{P_{N,j}} = \frac{\pi}{\sqrt{10005}}.}$$

10 Reproducibility map

The proof package contains the following certificates.

File	Exact obligation
`p28_full_closure_certificate.wl`	Optional independent Wolfram cross-check of the differential gauge and hypergeometric closure.
`p28_standalone_equations.py`	Mandatory dependency-free expansion of the four cleared Ore factorizations, the tail coefficient equations, the terminating base, generic, and top identities, the compact step polynomial, ascension, and the ${}_3F_2$ equation.
`p28_positive_cone.py`	Mandatory dependency-free verification of all sixteen positive transfer identities, all 285 positive numerator coefficients, the limiting matrix, and both official positive seed rows.
`p28_optimized_gauge.py`	Optional independent denominator-cleared replay of all sixteen gauge entries, including 176 scalar coefficient obligations.
`p28_dominant_product_algebra.py`	Preserved legacy alternative: dependency-free verification of $\mathcal{B}_m = \mathcal{S} + O(m^{-1})$, the characteristic polynomial, the root-separation inequalities, the left-eigenvector identity, and the four positive coordinate rewrites.
`p28_kernel_contiguity_certificate.sage`	Optional independent Sage check of the four displayed factorizations; it performs no Ore division.
`p28_lattice_hypotheses_certificate.sage`	Optional exact cross-check of the rank-one factorization, tail direction, and transformed ODE identities.
`p28_convergence_constants.py`	Exact rational verification of the coefficient bounds, $\alpha_{n+1}/\alpha_n > 64n^2$, $\\|\mathcal{B}_m\\|_\infty < 10000$, and $\beta(x_0) = 625/[R(R-1)] < 1$.
`all_four_columns_certificate.sage`	Optional Sage replay of the legacy balanced-limit and root algebra. The active proof uses Lemma 2.
`p28_parametric_pade_probe.py`	Diagnostic finite exact regression of the predicted valuations; it is not used as proof of an all-N statement.

The mandatory algebra checkers use only the standard library's `fractions.Fraction`, sparse coefficient dictionaries, and explicit addition, multiplication, differentiation, and coefficient comparison. They invoke no division algorithm, factorizer, root finder, special-function library, or numerical sample. Wolfram Language and SageMath are optional independent cross-checks, not a trust requirement.

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